# Analytical Benchmark for Criticality Uncertainty of Depleted Core Induced by Cross Section Perturbation

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## 1. Introduction

The nuclear data–induced criticality uncertainty of a depleted core is of interest in reactor design analysis, especially in the absence of experimental validation data [1],[2]. It consists of the direct contribution from nuclear data uncertainty and the indirect contribution from the number density uncertainty. To clearly understand the characteristics of criticality uncertainty with depletion, we propose a simple analytical benchmark for criticality with depletion, which provides closed-form solutions for number density, criticality, and their associated uncertainties based on first-order perturbation.

## 2. Analytical Benchmark for Criticality with Depletion Calculation

The benchmark is defined as a homogeneous system of volume $V$ with reflective boundary conditions. It consists of two artificial nuclides.

- nuclide 1: a pure fissioning nuclide whose fission yields produce only nuclide 2, with yield fraction $Y$.
- nuclide 2 : a pure neutron absorber, produced as the fission product of nuclide 1.

Both nuclides are assumed to have zero decay constants. The system is initially composed of entire nuclide 1 with density $N_0$.

The effective multiplication is expressed as:

$$k(t) = \frac{N_1(t)\nu\sigma_{f1}}{N_1(t)\sigma_{f1} + N_2(t)\sigma_{a2}} = \frac{G(t)}{D(t)}, \tag{1}$$

where $N_1$ and $N_2$ are number densities of nuclides 1 and 2, respectively, $\nu$ is the number of neutrons produced per fission, $\sigma_{f1}$ is the fission cross section (XS) of nuclide 1, and $\sigma_{a2}$ is the absorption XS of nuclide 2.

The number densities of nuclide 1 and 2 are governed by the following differential equations:

$$\frac{dN_1}{dt} = -N_1\sigma_{f1}\phi(t), \tag{2}$$

$$\frac{dN_2}{dt} = YN_1\sigma_{f1}\phi(t) - N_2\sigma_{a2}\phi(t), \tag{3}$$

with constant power normalization condition:

$$P = \kappa N_1(t)\sigma_{f1}\phi(t)V, \tag{4}$$

where $\phi$ is the neutron flux, $Y$ is the yield fraction of nuclide 2.

From power normalization condition, the flux is expressed as:

$$\phi(t) = \frac{P}{\kappa N_1(t)\sigma_{f1}V}. \tag{5}$$

Substituting Eq. (5) into Eqs. (2) and (3):

$$\frac{dN_1}{dt} = -A, \tag{6}$$

$$\frac{dN_2}{dt} = YA - \frac{Ar}{N_1}N_2, \tag{7}$$

where $A = P/\kappa V$ and $r = \sigma_{a2}/\sigma_{f1}$.

With this simplified system, analytical solutions for the time-dependent number densities, the effective multiplication factor, and their sensitivities to cross-section uncertainties can be derived, while preserving key characteristics of error propagation in criticality uncertainty with depletion.

By solving the differential equations, the number densities of nuclides 1 and 2 can be obtained as:

$$N_1(t) = N_0 - At, \tag{8}$$

$$N_2(t) = \begin{cases} YN_1 \log\left(\frac{N_0}{N_1}\right) & \text{if } r = 1, \\ -\frac{YN_1}{1-r} + \frac{YN_0}{1-r}\left(\frac{N_1}{N_0}\right)^r & \text{otherwise.} \end{cases} \tag{9}$$

Substituting Eqs. (8) and (9) into Eq. (1), the effective multiplication factor can be obtained as:

$$k(t) = \begin{cases} \dfrac{\nu}{1 + Y\log\left(\frac{N_0}{N_1}\right)} & \text{if } r = 1, \\ \dfrac{\nu}{1 - \frac{Yr}{1-r} + \frac{Yr}{1-r}\left(\frac{N_1}{N_0}\right)^{r-1}} & \text{otherwise.} \end{cases} \tag{10}$$

### 3. Criticality Uncertainty Analysis of Depleted Core

The criticality uncertainty of depleted core can be decomposed as direct contribution from XS and indirect contribution from the number density as:

$$(\Delta k)^2 = \left( \underbrace{\frac{\partial k}{\partial \sigma_{a2}}\bigg|_{N_2}}_{\text{direct}} + \underbrace{\frac{\partial k}{\partial N_2}\frac{\partial N_2}{\partial \sigma_{a2}}}_{\text{indirect}} \right)^2 (\Delta\sigma_{a2})^2 + \left( \underbrace{\frac{\partial k}{\partial \sigma_{f1}}\bigg|_{N_2}}_{\text{direct}} + \underbrace{\frac{\partial k}{\partial N_2}\frac{\partial N_2}{\partial \sigma_{f1}}}_{\text{indirect}} \right)^2 (\Delta\sigma_{f1})^2, \tag{11}$$

where $\Delta\sigma_{a2}$ and $\Delta\sigma_{f1}$ are XS perturbations. In Eq. (11), negative sensitivities are highlighted in red and positive sensitivities are highlighted in blue. The sign of sensitivities can be determined as follows:

$$\frac{\partial k}{\partial \sigma_{a2}}\bigg|_{N_2} = -\frac{G}{D^2}N_2 \le 0, \tag{12}$$

$$\frac{\partial k}{\partial \sigma_{f1}}\bigg|_{N_2} = \frac{\nu N_1 N_2 \sigma_{a2}}{D^2} \ge 0, \tag{13}$$

$$\frac{\partial k}{\partial N_2} = -\frac{G}{D^2}\sigma_{a2} < 0, \tag{14}$$

$$\frac{\partial N_2}{\partial \sigma_{a2}} = \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial \sigma_{a2}} = \frac{1}{\sigma_{f1}}\frac{\partial N_2}{\partial r} \le 0, \tag{15}$$

$$\frac{\partial N_2}{\partial \sigma_{f1}} = \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial \sigma_{f1}} = -\frac{\sigma_{a2}}{{\sigma_{f1}}^2}\frac{\partial N_2}{\partial r} \ge 0. \tag{16}$$

Even without mathematical analysis, the signs of Eqs. (12) to (16) can be understood from physical intuition. The signs of Eqs. (12) to (14) can be determined straightforwardly, while Eqs. (15) and (16), can be determined as follows:

$$\frac{\partial N_2}{\partial r} = \begin{cases} 0 & \text{if } r = 1, \\ \dfrac{Y N_0 x}{(1-r)^2} f(x,r) & \text{otherwise,} \end{cases} \tag{17}$$

where $x = N_1/N_0 \in (0,1]$ and

$$f(x,r) = -1 + x^{r-1} + (1-r)x^{r-1}\log(x). \tag{18}$$

By letting $z = (1-r)\log(x)$, Eq. (18) can be rewritten as:

$$f(z) = -1 + \exp(-z)(1+z), \tag{19}$$

$$f(z)\exp(z) = -\exp(z) + (1+z) \le 0, \tag{20}$$

where the inequality of Eq. (20) was derived from $\exp(z) \ge (1+z)$.

These results imply that the overall uncertainty shown in Eq. (11) is reduced by incorporating indirect contribution from the density perturbation which is induced by XS perturbation. Considering the XSs may be perturbed either by 1) nuclear data uncertainty or 2) Monte Carlo statistical uncertainty (when we use a Monte Carlo transport code), the indirect contribution from density perturbations acts as a clear negative feedback preventing the overall uncertainty from diverging as depletion proceeds.

To demonstrate the above inference, representative reactor parameters are selected as Table I. The three cases use the same parameters, except that Case 2 uses 1.5 times the absorption XS of Case 1 and Case 3 uses 1.5 times the fission XS of Case 1.

Table I. Parameters for case study

| Parameter | Units | Case 1 | Case 2 | Case 3 |
|---|---|---|---|---|
| $P/V$ | W/cm3 | 100 | | |
| $\kappa$ | MeV/fission | 200 | | |
| $\nu$ | - | 2 | | |
| $\sigma_{f1}$ | barn | 585 | 585 | 877.5 |
| $\sigma_{a2}$ | barn | 1000 | 1500 | 1000 |
| $\phi$ | n/cm2 | 1E+13 | | |
| $Y$ | - | 1.0 | | |
| $\Delta\sigma_{f1}$ | barn | 5.85 (1%) | 5.85 (1%) | 8.77 (1%) |
| $\Delta\sigma_{a2}$ | barn | 10 (1%) | 15 (1%) | 10 (1%) |

Figure 1 shows the criticality and number densities $N_1$ and $N_2$, while Figure 2 provides the criticality uncertainty as depletion proceeds. The accommodation of the indirect contribution from number density reduces the criticality uncertainty regardless of test cases. Furthermore, both Case 2 (larger absorption XS) and Case 3 (larger fission XS) leads to larger reduction of criticality uncertainty compared to Case 1 at the end of cycle (30 months).

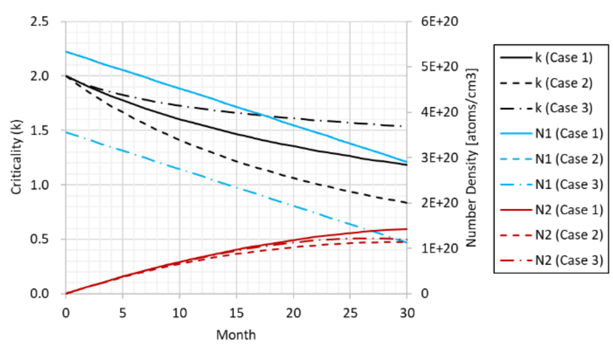


Figure 1. Criticality and number densities with depletion time.

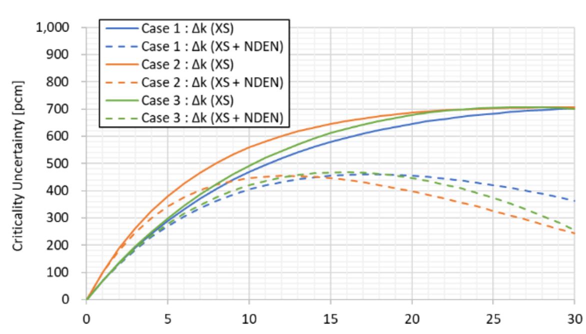


Figure 2. Criticality uncertainty with depletion time.

## 5. Summary

The simplified analytical benchmark was proposed to investigate the criticality uncertainty of depleted core and the analytic expression for the solutions and sensitivities were derived. Using this benchmark, we can understand the behavior of criticality uncertainty induced by the XS perturbations as depletion proceeds. A key finding is that the indirect contribution from density perturbations acts as a negative feedback, preventing the overall uncertainty from diverging during depletion. In addition, the XSs may be also perturbed either Monte Carlo statistical uncertainty when performing the Monte Carlo depletion calculation. The impact of this statistical uncertainty on criticality can be also mitigated by the negative feedback arising from the density perturbations.

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