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One Dimensional Inverse Heat Conduction Problem with the Walsh Method

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Abstract

The Walsh functions are used to obtain the space dependent thermal conductivity in one dimensional time dependent heat conduction medium. Although it is necessary to measure the temperatures in interior points, the Walsh reveals the real values without any numerical constraints. The proposed algorithm is quite different from other algorithms in that it is one-directional, that is, no iteration is necessary, and is numerically stable regardless of the functional characteristics of the thermal conductivity. This permits the estimation of yjermal conductivity when the medium consists of different materials.

1. Introduction

The inverse heat conduction problem (IHCP) is to estimate the spatially varying thermal conductivity of an inhomogeneous medium and is one of important issues in the engineering applications. And the algorithms in obtaining the solutions of IHCP are directly related to the similar problems encountered in various fields.

The IHCP is classified into two categories. One is the determination of thermal properties such as thermal conductivity and heat capacity, and the other is the estimation of boundary and initial conditions or heat sources. Each is based on the knowledge of temperatures or heat fluxes obtained through measurements. Regarding the former, some researchers have focused on the temperature dependent thermal properties [1-6], and others have dealt with the space dependent ones [7-8]. The measurements are taken at the interior points and/or on the boundary points of the heat conduction medium. As far as the practical application is concerned, the boundary measurements are more convenient than the interior measurements. In our previous study [9], we have shown that the solutions of the IHCP could be obtained with the boundary data only, contrary to Refs. [7] and [8], which used the interior and boundary data.

The algorithms of Ref. [9] in which the sensitivity function [6] is applied are composed of two phases, namely the forward problem and the inverse problem, and they are applied iteratively. In each phase, both the heat conduction equation and the sensitivity equation are solved respectively. The forward and inverse solutions are used to find the search step by the modified Newton Raphson method.

In general, the ill-posedness is encountered in the inverse problem and the regularization is required. The convergence is heavily dependent on the regularization factor as well as on the system model. Further, the conductivity is assumed to be at least first order derivative continuous function. This means that the system should be comprised of one material. For the case in which there are several heat conduction media, it is difficult to obtain the solution by the conventional approach. To resolve these problems, we take up a new approach by employing the Walsh function.

The Walsh function was initiated by Rademacher and independently developed by Walsh in the early 1920s. In recent years, the Walsh theory has been innovated and applied to various fields in engineering and sciences. One of the particular interests of the Walsh function is that the entire range of a function is modeled by a Fourier-type expansion instead of being created by piecewise integration from initial conditions with the possibility of straying from the path as errors accumulate. Even with the nonlinear equation, it is possible to implement an expansion approach because of the special properties of the Walsh function. Since Walsh functions have discontinuities built into their definition, they are especially suitable for approximating situations where a function changes rapidly in time or in position [10].

2. Walsh Functions and Integration/Differential Operators

An incomplete set of periodic rectangular orthonomal function was developed by Rademacher in 1922. The Rademacher functions, $r_k(t)$, have odd symmetry about origin and midpoint. This means that the set is incomplete since the sum of any number of the functions will have odd symmetry about these two points. It is not possible to expand a function which has even symmetry about the midpoint in a series of $r_k(t)$. The Rademacher functions have been combined by Walsh to form a complete orthonomal set of rectangular waves [11]. For an example, the Walsh matrix for the case of 2^n, n=3 is described in Eq. 1.

$$Wal(n=3)=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix} \quad (1)$$

The Walsh function has many off-springs such as Paley, Harr and Hadamard functions. They are different only in the combination method of the Rademacher functions and can be converted each other by use of the binary and gray numbers. All these functions have common properties of orthonomality and the Paley functions are used in this paper. For n=3, the Paley matrix is of eight (2^3) by eight matrix as :

$$Pal(n=3)=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}=\begin{pmatrix} Y_0(t) \\ Y_1(t) \\ \vdots \\ \vdots \\ \vdots \\ Y_7(t) \end{pmatrix} \quad (2)$$

The Walsh's family have important properties. First, it has the orthonomalty of

[illegible]

$$= 0, \quad m \neq n \quad (3)$$

And it is worth to note that the summation of each row or column vector is zero except the first ones. This implies that the Walsh function can be used as an efficient injection signal pattern in a various experiments, particularly in electric tomography.

Any function $f(t)$, $0 \le t \le 1$, can be expanded formally in a series of the form

$$f(t) = \sum_0 C_n Y_n(t) \text{ , where } C_n = \int_0 f(t) Y_n(t)\, dt \tag{4}$$

There is no strict constraint on the convergence. The function may be either continuous or not. However, as the number of expansion terms increases, the error becomes small, as in all other expansion functions.

The Walsh family can be used in the integration and differentiation of the given function very easily. For example, the integration of the zero-th mode of the Paley function in Eq. (2) is

$$\int_0^t Y_0(t')\, dt' = \int_0^t 1\, dt' = t = \begin{pmatrix} C_{0,0} & C_{0,1} & \cdots & C_{0,6} & C_{0,7} \end{pmatrix} \begin{pmatrix} Y_0(t) \\ Y_1(t) \\ \vdots \\ Y_6(t) \\ Y_7(t) \end{pmatrix} \tag{5}$$

It is the same for the rest modes, and the overall integration of the Paley functions is

$$\int_0^t Pal(t')dt' = \int_0^t \begin{pmatrix} Y_0(t') \\ Y_1(t') \\ \vdots \\ Y_6(t') \\ Y_7(t') \end{pmatrix} dt' = \begin{pmatrix} C_{0,0} & C_{0,1} & \cdots & C_{0,7} \\ C_{1,0} & C_{1,1} & \cdots & C_{1,7} \\ \vdots & \vdots & \vdots & \vdots \\ C_{7,0} & C_{7,1} & \cdots & C_{7,7} \end{pmatrix} \begin{pmatrix} Y_0(t) \\ Y_1(t) \\ \vdots \\ Y_6(t) \\ Y_7(t) \end{pmatrix} \equiv \boldsymbol{O} \cdot \boldsymbol{P}(t) \tag{6}$$

That is, the integration of the Paley function is described by itself with the integration operator matrix $\boldsymbol{O}$. For n =3, the integration operator matrix has the elements of

$$\boldsymbol{O} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & 0 & -\frac{1}{16} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & -\frac{1}{8} & 0 & -\frac{1}{16} & 0 & 0 \\ \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 \\ 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} \\ \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

Let's consider an arbitrary function $x(t)$, $0 \le t \le 1$. The Paley expansion of $x(t)$ up to $2^n - 1$ is

$$x(t) = X_0 Y_0(t) + X_1 Y_1(t) + \cdots + X_{2^n-2} Y_{2^n-2}(t) + X_{2^n-1} Y_{2^n-1}(t) = \boldsymbol{X} \cdot \boldsymbol{P}(t) \tag{8}$$

Let $y(t)$ be the integrated function of $x(t)$, and expand it by Paley. Then,

$$y(t) = \boldsymbol{Y} \cdot \boldsymbol{P}(t) \tag{9}$$

where $\boldsymbol{Y}$ is the coefficient vector. Since $y(t) = \int_0 x(t')\, dt'$,

$$y(t) = \boldsymbol{Y} \cdot \boldsymbol{P}(t) = \boldsymbol{X} \int_0 \boldsymbol{P}(t')\, dt' = \boldsymbol{X} \cdot \boldsymbol{O} \cdot \boldsymbol{P}(t) \tag{10}$$

Hence, any function can be integrated through the multiplication of its coefficient vector by the integration operator.

The differentiation is the inverse procedure of the integration and is obtained by the operation of

$$\frac{dy(t)}{dt} = \boldsymbol{X} \cdot \boldsymbol{O}^{-1} \cdot \boldsymbol{P}(t) \tag{11}$$

3. Application to the Inverse Heat Conduction Problem

The one dimensional time dependent heat conduction is

$$\frac{\partial}{\partial x} k(x) \frac{\partial T(x,t)}{\partial x} = \frac{\partial T(x,t)}{\partial t} \tag{12}$$

For convenience, the density and heat capacity are assumed as unity. The problem is posed to determine the spatial distribution of thermal conductivity with the measurement data of $T(x,t)$.

By letting $\frac{\partial T(x,t)}{\partial t} = g(x,t)$, Eq.(12) is

$$\frac{\partial}{\partial x}\left(k(x) \frac{\partial}{\partial x} T(x,t) \right) = g(x,t) \tag{13}$$

At a given moment, *T(x,t)* and its time derivatives are known from the measurement, and Eq. (13) can be written as

$$\frac{d}{dx}\left(k(x) \frac{dT(x)}{dx} \right) = g(x) \tag{14}$$

Each function of above equation is expanded by Paley as

$$k(x) = \boldsymbol{K} \cdot \boldsymbol{P}(x), \quad T(x) = \boldsymbol{T} \cdot \boldsymbol{P}(x), \quad g(x) = \boldsymbol{G} \cdot \boldsymbol{P}(x) \tag{15}$$

Then

$$\frac{dT(x)}{dx} = \boldsymbol{T} \cdot \boldsymbol{O}^{-1} \cdot \boldsymbol{P}(x) = \boldsymbol{T}_1 \cdot \boldsymbol{P}(x) \tag{16}$$

And Eq. (14) has the form of

$$\frac{d}{dx}\left((\boldsymbol{K} \cdot \boldsymbol{P}(x)) \circ (\boldsymbol{T}_1 \cdot \boldsymbol{P}(x)) \right) = \boldsymbol{G} \cdot \boldsymbol{P}(x) \tag{17}$$

where the operator $\circ$ indicates the multiplication of element by element.

By integrating Eq. (17), the thermal conductivity is obtained from

$$k(x) = \boldsymbol{K} \cdot \boldsymbol{P}(x) = \frac{\int_0^x \boldsymbol{G} \cdot \boldsymbol{P}(x) dx}{\boldsymbol{T}_1 \cdot \boldsymbol{P}(x)} + \frac{(\boldsymbol{K} \cdot \boldsymbol{P}(0))(\boldsymbol{T}_1 \cdot \boldsymbol{P}(0))}{\boldsymbol{T}_1 \cdot \boldsymbol{P}(x)} = \frac{\boldsymbol{G} \cdot \boldsymbol{O} \cdot \boldsymbol{P}(x)}{\boldsymbol{T}_1 \cdot \boldsymbol{P}(x)} + \frac{(\boldsymbol{K} \cdot \boldsymbol{P}(0))(\boldsymbol{T}_1 \cdot \boldsymbol{P}(0))}{\boldsymbol{T}_1 \cdot \boldsymbol{P}(x)} \tag{18}$$

The equation above requires the boundary condition at $x = 0$. If the boundary value at $x = 1$ is known, the boundary value at $x = 0$ can be found easily by the property of orthonormality. If no boundary conditions are given, there are one more unknowns than the matrix size, and even in this case, the equation can be solved by the pseudo inversing.

Figures 1 and 2 show the validity of the Walsh (Paley) approach in obtaining the thermal conductivity in IHCP. The number of nodes is $2^5 = 32$, including the both ends. At the initial state, the temperatures are assumed to be all the same. As a transient condition, the boundary heat fluxes are given at both ends as $q(0) = q(1) = 10$. The thermal conductivities of all the nodes are arbitrarily set to 2. The measurement is assumed to be made 20 times and to be terminated at $t = 1 sec$.

In Fig. 1, the real value of the thermal conductivity is $k(x) = |sin(2p\, x)|$. The results of calculation show a good approximation. It is worth to note that the thermal conductivity is continuous at the midpoint, but its derivatives are not. This is different from our previous study [9] in which the conductivity was assumed to be the first order derivative function. Figure 2 describes the calculated values against the real values of the conductivity. The real one is $k(x) = |sin(2p\, x)|$, $0 \le x \le 0.5$ and $k(x) = x^2 + 1$, $0.5 < x \le 1$. Even in this case in which the conductivity is discontinuous, let alone its derivatives, the calculated values follow the real values.

It is to be noted that no iteration is necessary in the proposed approach. As time goes on, the conductivities converge from the initially assumed values to the real ones. And the calculated values are almost the same after 8-10 times of calculation, that is, around the t = 0.4~0.5 sec even though the transient continues. The deviations between the real and estimated values are only dependent on the bit numbers, not on the algorithms. Therefore, the numerical process is very stable, which is one of the attractions of the Walsh.

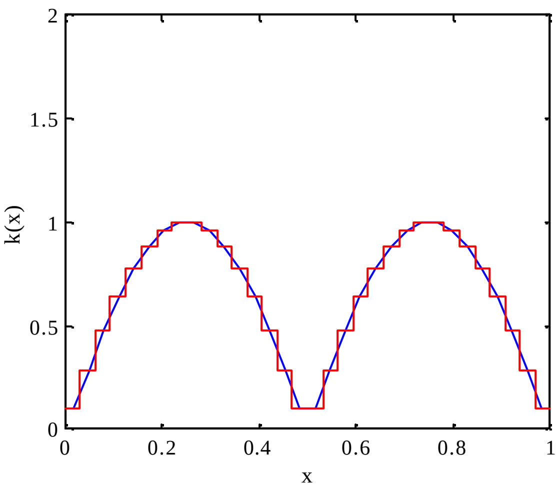

Fig. 1 The estimation of thermal conductivity $k(x) = |sin(2p\, x)|$

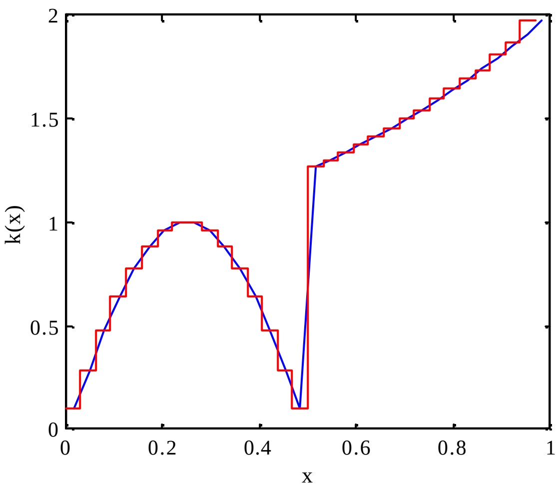

Fig. 2 The estimation of thermal conductivity $k(x) = |sin(2p\, x)|$, $0 \le x \le 0.5$
$k(x) = x^2 + 1$, $0.5 < x \le 1$

4. Conclusion

The Walsh functions are used to obtain the space dependent thermal conductivity in one dimensional time dependent heat conduction medium. Although it is necessary to measure the temperatures in interior points, the Walsh reveals the real values without any numerical difficulties. The proposed algorithm is quite different from other algorithms in that it is one-directional, that is, no iteration is necessary, and is numerically stable regardless of the functional characteristics of the thermal conductivity. This permits the estimation of thermal conductivity when the medium consists of different materials.

One of the important characteristics of the Walsh function is that the sum of row element or column element is zero, except the first row or column vector. This property hints that the injection signal might have a Walsh function form when it is required that the summation of the injection signals should be zero. For example, in the electric tomography apparatus, the image reconstruction is dependent on the injection signal, and the Walsh wave might be ideal for such a case.

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