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A Spectral Analytic Discrete-Ordinates Transport Method Based on Infinite Medium Green's Function for Multiplying Fission Source Problems

Dae Han Jung and Nam Zin Cho
Korea Advanced Institute of Science and Technology
Department of Nuclear Engineering
373-1 Kusong-dong, Yusong-gu
Taejon, Korea 305-701
Email : nzcho@mail.kaist.ac.kr

Abstract

Analytic solutions of the multigroup discrete ordinates transport equation with linearly anisotropic scattering and fission source for multi-layered slab problems are obtained by using the infinite medium Green's function (IMGF) and Placzek's lemma. In this approach, the infinite medium Green's function is derived analytically by using the spectral analysis for the multigroup discrete ordinates transport equation and its transposed equation, and this infinite medium solution is related to the finite medium solution by Placzek's lemma. In eigenvalue problems having fission source, complex eigenvalues can occur. As such equations involve the k eigenvalue as a non-linear parameter, to obtain criticality Newton's chord method combined with bisection is used. The resulting equation leads to an exact relation that represents the outgoing angular fluxes in terms of the incoming angular fluxes and fission source for each slab. For heterogeneous problems having multi-layered slabs, the slabs are coupled through the interface angular fluxes. Since all derivations are performed analytically, the method gives exact solution with no truncation error. After the interface angular fluxes are calculated by using an iterative method, the continuous spatial distribution of the angular flux (i.e.analytic solution) in each slab is given straightforwardly in terms of the IMGF and the boundary angular fluxes. Therefore, in our method, the number of meshes that is equal to the number of the homogeneous slabs is sufficient.

I. Introduction

Since the neutron transport equation cannot be solved analytically even in slab geometry, much attention has been given to the problem of obtaining accurate numerical methods of the transport equation. The discrete ordinates approximation of angular variable and the multigroup approximation of energy variable are the most direct approach to simplify the complexities of the transport equation. Recently, some authors have devised exact solution methods with no truncation error for solving the multigroup discrete ordinates problems in slab geometry. First, Barros and Larsen[2][3] have developed the spectral Green's function(SGF) method where an exact relation between cell-edge and cell-average angular fluxes is derived by using a spectral analysis (i.e., obtaining eigenfunctions). Second is a direct method by using the Laplace transform[4] and its inverse transform.

In this paper, a new method that gives analytic solutions (with no spatial truncation error) of the slab geometry multigroup discrete ordinates transport equation with linearly anisotropic scattering and multiplying fission source is presented. This is an extension of the one group method[5] by Hong and Cho. The method is based on the infinite medium Green's function and Placzek's lemma[6]. The IMGF is derived analytically by using the spectral analysis for the multigroup discrete ordinates transport equation and its transposed equation. The infinite medium solution is related to the finite medium solution through the Placzek's lemma. This procedure gives an exact relation that represents the outgoing angular fluxes in terms of the incoming angular fluxes and fission source for each slab. In multi-slab problems, the slabs are simply coupled through the interface angular fluxes. Therefore, the interface angular fluxes are obtained, the continuous distribution of the angular flux in each slab are calculated straightforwardly with IMGF.

Our method is new in the area of the discrete ordinates transport methods. In case of the continuous angle, Case[7] originally obtained IMGF of the one group transport equation with singular eigenfunctions while the evaluation of IMGF is difficult and numerical results were not readily available. Second, the Case's IMGF has been used in the F_N[8] and C_N[9] methods that gives very accurate solutions. In case of multigroup problems, the IMGF is obtained by using the Fourier transform[9] and its inversion rather than Case's methodology. A similar approach by Ganapol[10][11] was recently performed to obtain highly accurate solution (analytic benchmark solution) of the one group transport equation. However, to our knowledge, IMGF of the slab geometry multigroup discrete ordinates transport equation with fission source has not been derived and used to solve multi-slab problems. Therefore, the key feature of our approach is the analytical derivation of IMGF of the multigroup slab geometry discrete ordinates transport equation with linearly anisotropic scattering and multiplying fission source problem and its effective use in generating the analytic solutions for multigroup multi-slab problem. In eigenvalue problems having fission source, purely complex eigenvalues can occur. As such equations involve the k eigenvalue as a non-linear parameter, to obtain criticality Newton's chord method[12] with bisection is used.

II. Theory and Methodology

II.1. Spectral Analysis

The multigroup slab geometry discrete ordinates transport equation with linearly anisotropic scattering and fission source is described by

$$\mu_m \frac{d\vec{\psi}_m(x)}{dx} + \Sigma\vec{\psi}_m(x) = \Sigma_{s0}\vec{\phi}(x) + 3\mu_m\Sigma_{s1}\vec{\phi}_1(x) + \frac{1}{k_{eff}}\nu\sigma_f\vec{\phi}(x), \tag{1}$$

where the vector $\vec{\psi}_m$ has components $\psi_{g,m}$ which are the angular fluxes for each energy group g and direction m, the vector $\vec{\phi}$ has components ϕ_g which are the scalar fluxes for each energy group g and the vector $\vec{\phi}_1$ has components $\phi_{1,g}$ which are the net currents for each energy group g. In Eq.(1), the scattering matrix(Σ_{s0}) that represents the isotropic component of the scattering. The scattering matrix(Σ_{s1}) that represents the linearly anisotropic component of the scattering can be similarly given and the diagonal matrix Σ has components that are the total macroscopic cross sections for each energy group. In addition, σ_f is the group fission cross section, ν is the average number of prompt neutrons emitted per fission and k is an eigenvalue. We will perform a spectral analysis of the S_N equation Eq.(1) to obtain the eigenfunction, Eq.(1) is considered :

$$
\begin{aligned}
\mu_m \frac{d\vec{\psi}_m(x)}{dx} + \Sigma\vec{\psi}_m(x) &= (\Sigma_{s0} + \frac{1}{k_{eff}}\nu\sigma_f)\vec{\phi}(x) + 3\mu_m\Sigma_{s1}\vec{\phi}_1(x) \\
&= \Sigma_{s0}^*\vec{\psi}_m(x) + 3\mu_m\Sigma_{s1}\vec{\phi}_1(x),
\end{aligned} \tag{2}
$$

where Σ_{so}^* is combined fission cross section term and isotropic component of the scattering. We seek solution of the following form [13] :

$$
\vec{\psi}_m(x) = e^{-x/\nu}\vec{\phi}_\nu(\mu_m). \tag{3}
$$

Substituting Eq.(3) into Eq.(2) gives

$$
(-\frac{\mu_m}{\nu}\mathbf{I} + \Sigma)\vec{\phi}_\nu(\mu_m) = \Sigma_{s0}^*\vec{N}_0(\nu) + 3\mu_m\Sigma_{s1}\vec{N}_1(\nu), \tag{4}
$$

where $\vec{N}_0(\nu) = \sum_{m=1}^{N}\omega_m\vec{\phi}_\nu(\mu_m)$ and $\vec{N}_1(\nu) = \sum_{m=1}^{N}\omega_m\mu_m\vec{\phi}_\nu(\mu_m)$.

After some algebraic procedures with Eq.(4), the following eigenvectors are obtained :

$$
\begin{aligned}
\vec{N}_0(\nu) &= \sum_{m=1}^{N}\omega_m(-\frac{\mu_m}{\nu}\mathbf{I} + \Sigma)^{-1}\Sigma_{s0}^*\vec{N}_0(\nu) \\
&\quad + 3\sum_{m=1}^{N}\omega_m\mu_m(-\frac{\mu_m}{\nu}\mathbf{I} + \Sigma)^{-1}\Sigma_{s1}\vec{N}_1(\nu), \\
\vec{N}_1(\nu) &= \nu(\Sigma - \Sigma_{s0}^*)\vec{N}_0(\nu).
\end{aligned} \tag{5}
$$

The eigenvalues ν are determined by the following characteristic equation :

$$
\begin{aligned}
\det[\mathbf{I} &- \sum_{m=1}^{N}\omega_m(-\frac{\mu_m}{\nu}\mathbf{I} + \Sigma)^{-1}\Sigma_{s0}^* \\
&- 3\nu\sum_{m=1}^{N}\omega_m\mu_m(-\frac{\mu_m}{\nu}\mathbf{I} + \Sigma)^{-1}\Sigma_{s1}\nu(\Sigma - \Sigma_{s0}^*)] = 0,
\end{aligned} \tag{6}
$$

where *det* means the determinant of a matrix. Eq.(6) is a polynomial of $G \times N$'th degree and its roots(ν) are the eigenvalues of the G-group discrete ordinates problem (i.e., Eq.(2)). The roots are symmetrically distributed around the origin due to symmetry of the Gauss Legendre quadrature set. In practical problem, eigenvalue problems having fission source, the eigenvalues consists of real and purely complex.

Since the orthogonality of the eigenvectors was efficiently used in analytically deriving the IMGF of the one group problems, a similar property is devised in the multigroup problem. However, the bi-orthogonality [14] between the eigenvector of the forward problem and eigenvector of the transposed problem has been found in the continuous angle case. In this paper, the bi-orthogonality is derived in a similar fashion for the discrete ordinates transport problems. The spectral analysis of the transposed equation of Eq.(2) can be performed similarly as the forward case. The result is given by

$$(-\frac{\mu_m}{\nu}\mathbf{I} + \Sigma)\vec{\phi}^*_\nu(\mu_m) = \Sigma^{*\mathbf{T}}_{s0}\vec{N}^*_0(\nu) + 3\mu_m\Sigma^{\mathbf{T}}_{s1}\vec{N}^*_1(\nu), \tag{7}$$

Using Eq.(4) and Eq.(7) gives the following bi-orthogonality :

$$\sum_{m=1}^{N} \omega_m\mu_m < \vec{\phi}^*_{\nu_\beta}(\mu_m), \vec{\phi}_{\nu_\alpha}(\mu_m) >= M_{\nu_\alpha}\delta_{\alpha\beta}, \tag{8}$$

where $\delta_{\alpha\beta}$ is the Kronecker's delta, $< \cdot >$ means the dot product.

To evaluate the transposed angular eigenvector $(\vec{\phi}^*(\mu_m))$, the transposed eigenvectors $(\vec{N}^*_0, \vec{N}^*_1)$ must be evaluated. The evaluation of the transposed eigenvectors is similar to that of the forward eigenvectors $(\vec{N}_0, \vec{N}_1)$ and omitted here.

II.2. Infinite Medium Green's Function (IMGF)

An infinite homogeneous medium having a unit source at origin, emitting in the direction of μ_p and with a particular energy of g_p is considered. The solution of this infinite medium problem is called the infinite medium Green's function $\vec{G}^{g_p}(0,\mu_p;x,\mu_m)$. The problem is described as follow :

$$\begin{aligned}(\mu_m\frac{d}{dx} + \Sigma)\vec{G}^{g_p}(0,\mu_p;x,\mu_m) &= \Sigma^*_{s0}\sum_{m=1}^{N}\omega_n\vec{G}^{g_p}(0,\mu_p;x,\mu_m) \\ &+ \Sigma_{s1}(3\mu_m)\sum_{m=1}^{N}\omega_n\mu_m\vec{G}^{g_p}(0,\mu_p;x,\mu_m) + \delta(x)\delta(\mu_m-\mu_p)\vec{\delta}_{g_p},\end{aligned} \tag{9}$$

where the vector $\vec{\delta}_{g_p}$ has only one non-zero component of unity at the g_p'th position, and the $\delta(\mu_m - \mu_p)$ is defined to satisfy $\sum_{m=1}^{N}\omega_m\delta(\mu_m - \mu_p) = 1$. $G^{g_p\to g}(0,\mu_p;x,\mu_m)$ represents the angular flux at x for direction μ_m of energy group g due to the unit source at origin for direction μ_p of energy group g_p.

For position $x \neq 0$, the solution must satisfy the homogeneous equation and the following condition. The first condition is the finiteness at infinity. The second condition that is used to determine the expansion coefficients is the jump condition at the source position. This condition is derived by integrating Eq.(9) over an infinitesimal interval around the source position :

$$\vec{G}^{g_p}(0,\mu_p;o^+,\mu_m) - \vec{G}^{g_p}(0,\mu_p;o^-,\mu_m) = \frac{\delta(\mu_m-\mu_p)}{\mu_m}\vec{\delta}_{g_p}. \tag{10}$$

With these conditions and bi-orthogonality of the eigenvectors, we obtain the final result of the IMGF with the unit source at $x = x_0$ is given by

$$\vec{G}^{g_p}(x_0, \mu_p; x, \mu_m) = \begin{cases} \displaystyle\sum_{\alpha=1}^{NG/2} e^{-(x-x_0)/\nu_\alpha} \frac{\left[\vec{\phi}^*_{\nu_\alpha}(\mu_p)\right]_{g_p}}{M_{\nu_\alpha}} \vec{\phi}_{\nu_\alpha}(\mu_m), & x > x_0, \\ \displaystyle -\sum_{\alpha=NG/2+1}^{NG} e^{-(x-x_0)/\nu_\alpha} \frac{\left[\vec{\phi}^*_{\nu_\alpha}(\mu_p)\right]_{g_p}}{M_{\nu_\alpha}} \vec{\phi}_{\nu_\alpha}(\mu_m), & x < x_0. \end{cases} \tag{11}$$

where $[\vec{a}]_p$ represents the p'th component of a vector $\vec{a}$.

II.3. Computational Scheme

In this section, the computational method using IMGF for obtaining the analytic solution of the multigroup slab geometry discrete ordinates transport equation is derived. First, consider a homogeneous finite slab problem with given incoming angular fluxes at boundaries:

$$\begin{gathered} \mu_m \frac{d\vec{\psi}_m(x)}{dx} + \Sigma\vec{\psi}_m(x) = \Sigma^*_{s0}\vec{\psi}_m(x) + 3\mu_m\Sigma_{s1}\vec{\phi}_1(x), \quad x \in [-a, a], \\ \vec{\psi}_m(-a) = \vec{\psi}^{in}_{L,m}, \quad \mu_m > 0, \quad \vec{\psi}_m(a) = \vec{\psi}^{in}_{R,m}, \quad \mu_m < 0, \end{gathered} \tag{12}$$

where a is the half thickness of the slab. We note that the above finite medium solution $\vec{\psi}_m(x)$ can be represented in terms of an infinite medium solution $[\vec{\psi}^{\infty}_m(x)]$ due to the Placzek's lemma as follows :

$$\begin{gathered} H_*(x)\vec{\psi}_m(x) = \vec{\psi}^{\infty}_m, \quad H_*(x) = \begin{cases} 1, & x \in [-a, a], \\ 0, & \text{otherwise}, \end{cases} \\ \mu_m \frac{d\vec{\psi}^{\infty}_m(x)}{dx} + \Sigma\vec{\psi}^{\infty}_m(x) = \Sigma^*_{s0}\vec{\psi}^{\infty}_m(x) + 3\mu_m\Sigma_{s1}\vec{\phi}^{\infty}_1(x) + \mu_m\vec{\psi}_m(x)[\delta(x+a) - \delta(x-a)], \end{gathered} \tag{13}$$

where $H_*(x)$ is a step function and $x \in [-\infty, \infty]$. Since IMGF is available, the solution for the finite solution can be given by

$$\psi_{p,m}(x) = \sum_{q=1}^{G}\sum_{n=1}^{N} \omega_n G^{q\to p}(-a, \mu_n; x, \mu_m)\mu_n\psi_{q,n}(-a) - \sum_{q=1}^{G}\sum_{n=1}^{N} \omega_n G^{q\to p}(a, \mu_n; x, \mu_m)\mu_n\psi_{q,n}(a), \tag{14}$$

where the indices p and q were used to represent energy group. In Eq.(13), it must be noted that the finite solution is equal to the infinite medium solution for $x \in [-a, a]$. Inserting $x = a$ into Eq.(14) and separating the boundary angular fluxes into the incoming and outgoing parts give

$$\begin{aligned} \psi_{p,m}(a) = & \sum_{q=1}^{G}\sum_{n=1}^{N/2} \omega_n G^{q\to p}(-a, \mu_n; a, \mu_m)\mu_n\psi^{in}_{L,q,n} - \sum_{q=1}^{G}\sum_{n=N/2+1}^{N} \omega_n G^{q\to p}(a, \mu_n; a^-, \mu_m)\mu_n\psi^{in}_{R,q,n} \\ & + \sum_{q=1}^{G}\sum_{n=N/2+1}^{N} \omega_n G^{q\to p}(-a, \mu_n; a, \mu_m)\mu_n\psi^{out}_{L,q,n} - \sum_{q=1}^{G}\sum_{n=1}^{N/2} \omega_n G^{q\to p}(a, \mu_n; a^-, \mu_m)\mu_n\psi^{out}_{R,q,n}. \end{aligned} \tag{15}$$

Multiplying Eq.(15) with $\mu_m[\phi^*_{\nu_\alpha}(\mu_m)]_p$ and summing over m and p lead to

$$\begin{aligned}
&\sum_{m=1}^{N/2}\omega_m\mu_m < \vec{\phi}^*_{\nu_\alpha}(\mu_m), \vec{\psi}^{out}_{R,m} > + e^{-2a/\nu_\alpha}\sum_{m=1}^{N/2}\omega_m\mu_m < \vec{\phi}^*_{\nu_\alpha}(-\mu_m), \vec{\psi}^{out}_{L,m} > \\
&= \sum_{m=1}^{N/2}\omega_m\mu_m < \vec{\phi}^*_{\nu_\alpha}(-\mu_m), \vec{\psi}^{in}_{R,m} > + e^{-2a/\nu_\alpha}\sum_{m=1}^{N/2}\omega_m\mu_m < \vec{\phi}^*_{\nu_\alpha}(\mu_m), \vec{\psi}^{in}_{L,m} > \\
&\qquad\qquad \alpha = 1,2,3,...,NG/2
\end{aligned} \tag{16}$$

where $< \cdot >$ means the dot product. In deriving Eq.(16), the bi-orthogonality of the eigenvectors was used. It must be noted that Eq.(16) can easily treat arbitrarily fission source. Similarly, the corresponding equation for $x = -a$ can be derived. In case of real eigenvalue equal to fixed source problem.[1] So we must give attention to purely complex eigenvalues. If we substituting complex eigenvalue into two equations, its are in agreement with real equations. Because two equations relate to precisely complex conjugate. If ν_p is complex eigenvalue($\nu_p = i\lambda_p$), we can write as following

$$\vec{\phi}^*_{\nu_p}(\mu_m) = \mathrm{Re}[\vec{\phi}^*_{\nu_p}(\mu_m)] + i\mathrm{Im}[\vec{\phi}^*_{\nu_p}(\mu_m)], \tag{17}$$

where λ_p is real. Substituting Eq.(17) into Eq.(16), we can obtain real part and imaginary part. Real part as follow :

$$\begin{aligned}
&\sum_{m=1}^{N/2}\omega_m\mu_m < \mathrm{Re}[\vec{\phi}^*_{\nu_p}(\mu_m)], \vec{\psi}^{out}_{R,m} > + cos\frac{2a}{\lambda_p}\sum_{m=1}^{N/2}\omega_m\mu_m < \mathrm{Re}[\vec{\phi}^*_{\nu_p}(\mu_m)], \vec{\psi}^{out}_{L,m} > \\
&+ sin\frac{2a}{\lambda_p}\sum_{m=1}^{N/2}\omega_m\mu_m < \mathrm{Im}[\vec{\phi}^*_{\nu_p}(\mu_m)], \vec{\psi}^{out}_{L,m} > \\
&= \sum_{m=1}^{N/2}\omega_m\mu_m < \mathrm{Re}[\vec{\phi}^*_{\nu_p}(\mu_m)], \vec{\psi}^{in}_{R,m} > + cos\frac{2a}{\lambda_p}\sum_{m=1}^{N/2}\omega_m\mu_m < \mathrm{Re}[\vec{\phi}^*_{\nu_p}(\mu_m)], \vec{\psi}^{in}_{L,m} > \\
&- sin\frac{2a}{\lambda_p}\sum_{m=1}^{N/2}\omega_m\mu_m < \mathrm{Im}[\vec{\phi}^*_{\nu_p}(\mu_m)], \vec{\psi}^{in}_{L,m} >,
\end{aligned} \tag{18}$$

where imaginary part as follow :

$$\begin{aligned}
&\sum_{m=1}^{N/2}\omega_m\mu_m < \mathrm{Im}[\vec{\phi}^*_{\nu_p}(\mu_m)], \vec{\psi}^{out}_{R,m} > - cos\frac{2a}{\lambda_p}\sum_{m=1}^{N/2}\omega_m\mu_m < \mathrm{Im}[\vec{\phi}^*_{\nu_p}(\mu_m)], \vec{\psi}^{out}_{L,m} > \\
&+ sin\frac{2a}{\lambda_p}\sum_{m=1}^{N/2}\omega_m\mu_m < \mathrm{Re}[\vec{\phi}^*_{\nu_p}(\mu_m)], \vec{\psi}^{out}_{L,m} > \\
&= -\sum_{m=1}^{N/2}\omega_m\mu_m < \mathrm{Im}[\vec{\phi}^*_{\nu_p}(\mu_m)], \vec{\psi}^{in}_{R,m} > + cos\frac{2a}{\lambda_p}\sum_{m=1}^{N/2}\omega_m\mu_m < \mathrm{Im}[\vec{\phi}^*_{\nu_p}(\mu_m)], \vec{\psi}^{in}_{L,m} > \\
&+ sin\frac{2a}{\lambda_p}\sum_{m=1}^{N/2}\omega_m\mu_m < \mathrm{Re}[\vec{\phi}^*_{\nu_p}(\mu_m)], \vec{\psi}^{in}_{L,m} >,
\end{aligned} \tag{19}$$

these equations can be written in the following vector form :

$$\mathbf{A}\vec{\psi}^{out} = \mathbf{B}\vec{\psi}^{in}, \tag{20}$$

where $\vec{\psi}^{out}$ and $\vec{\psi}^{in}$ consist of the cell-edge outgoing and incoming angular fluxes for the finite slab respectively, and both $\mathbf{A}$ and $\mathbf{B}$ are $GN \times GN$ matrices.

The problems having heterogeneous materials (consisting of multilayered homogeneous slabs) can be solved by an iterative scheme on the interface angular fluxes (e.g., red-black iteration with one-node block inversion) with the continuity of the interface angular fluxes. Finally, if all the cell-edge angular fluxes are calculated, the analytic distribution for each homogeneous slab is easily calculated by Eq.(11) and Eq.(14). The resulting equation for the case of fission and isotropic source is explicitly given by

$$\begin{aligned}\psi_{p,m}(x) = & \sum_{\alpha=1}^{NG/2} \frac{1}{M_{\nu_\alpha}} e^{-\frac{(x+a)}{\nu_\alpha}} [\vec{\phi}_{\nu_\alpha}(\mu_m)]_p \sum_{q=1}^{G}\sum_{n=1}^{N} \omega_n \mu_n [\vec{\phi}^*_{\nu_\alpha}(\mu_m)]_q \psi_{L,q,n} \\ & - \sum_{\alpha=1}^{NG/2} \frac{1}{M_{\nu_\alpha}} e^{\frac{(x-a)}{\nu_\alpha}} [\vec{\phi}_{-\nu_\alpha}(\mu_m)]_p \sum_{q=1}^{G}\sum_{n=1}^{N} \omega_n \mu_n [\vec{\phi}^*_{-\nu_\alpha}(\mu_m)]_q \psi_{R,q,n}.\end{aligned} \tag{21}$$

Since all derivations except the discrete ordinates approximation are analytic, Eq.(21) is the analytic solution of the multigroup discrete ordinates transport equation in slab geometry. For the case of general source without uniform and isotropic assumptions, the analytic solution can be also written down but omitted here. In Eq.(21), it must be noted that this equation has both the incoming angular fluxes and the outgoing angular fluxes at the boundaries of a homogeneous slab. If only region averaged scalar fluxes are required, the use of the balance equations is more convenient than the use of Eq.(21).

III. Scalar Flux and Multiplication Factor k Calculation

From the multigroup slab geometry discrete ordinates transport equation with linearly anisotropic scattering and fission source, we can derived scalar flux as

$$\vec{\phi}_i = \frac{1}{h_i}(\Sigma - \Sigma^*_{s0})^{-1}(\vec{L}^{in}_{i-\frac{1}{2}} + \vec{L}^{in}_{i+\frac{1}{2}}), \tag{22}$$

where

$$\begin{aligned}\vec{L}^{in}_{i-\frac{1}{2}} &= \sum_{\mu_m<0} \omega|\mu_m|\vec{\psi}_{m,i+\frac{1}{2}} - \sum_{\mu_m>0} \omega|\mu_m|\vec{\psi}_{m,i+\frac{1}{2}}, \\ \vec{L}^{in}_{i+\frac{1}{2}} &= \sum_{\mu_m>0} \omega|\mu_m|\vec{\psi}_{m,i-\frac{1}{2}} - \sum_{\mu_m<0} \omega|\mu_m|\vec{\psi}_{m,i-\frac{1}{2}}.\end{aligned} \tag{23}$$

We can change the matrix structure. So we modify Eq.(20) by using LU decomposition.

$$\vec{\psi}^{out} = H(k_{eff})\vec{\psi}^{in}. \tag{24}$$

To improve on the k_{eff} value convergence, we apply the method by Maiani and Montagnini [12] . In Fig.1, we can make a coupling iteration scheme. Then we can make a matrix with

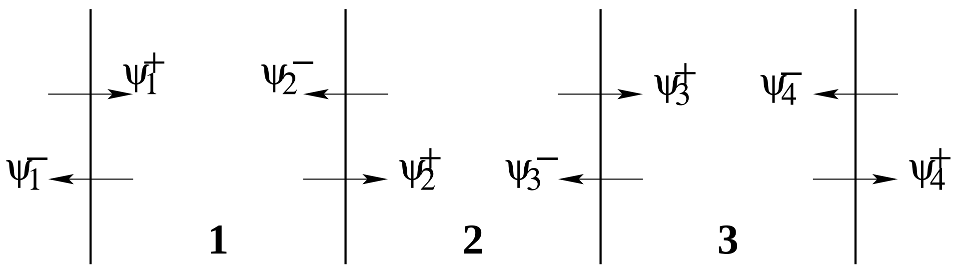


Figure 1: Coupling iteration

incoming and outgoing angular flux term :

$$\begin{pmatrix} \psi_1^+ \\ \psi_2^- \\ \psi_2^+ \\ \psi_3^- \\ \psi_3^+ \\ \psi_4^- \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} \psi_1^- \\ \psi_2^+ \\ \psi_2^- \\ \psi_3^+ \\ \psi_3^- \\ \psi_4^+ \end{pmatrix}, \tag{25}$$

which can be written in the following vector form :

$$\vec{\psi}^{in} = M \cdot \vec{\psi}^{out}, \tag{26}$$

where M means a coupling matrix. a is determined by boundary condition. From Eq.(24) and Eq.(26), we obtain

$$\psi^{in} = MH(k)\psi^{in}. \tag{27}$$

Nontrivial solutions of Eq.(27) do exist, but only in correspondence with particular values of the parameter k_{eff} involved in the $H(k)$ matrix elements. We have, in fact, a non-linear eigenvalue problem

$$\psi^{in} = \Theta(k)\psi^{in}, \tag{28}$$

where $\Theta(k) = MH(k)$. The solution technique consists in considering k_{eff} as a control parameter for the following auxiliary linear eigenvalue problem :

$$\alpha\psi^{in} = \Theta(k)\psi^{in}, \tag{29}$$

and looking for those values of k_{eff} which are such that an α-eigenvalue equals unity.

To continue the argument we shall assume that the partial currents are given in terms of point values. It is thus clear that the matrix $\Theta(k)$ is non-negative (this follows directly from the physical meaning of its elements). Hence, by the theorem of Frobenius, $\Theta(k)$ has a positive eigenvalue α, the fundamental eigenvalue, which exceeds the real part of any other eigenvalue. To α there corresponds a unique, non-negative fundamental eigenvector ψ^{in}. Thus we may limit ourselves to the fundamental eigenvalue and eigenvector and simply look for the roots of the transcendental equation

$$\alpha(k) = 1. \tag{30}$$

As $\alpha(k)$ turns out to be a monotonic function of k, Eq.(30) has a unique root, which corresponds to the value of the effective multiplication constant for the system under study. The research for α is performed by the following iterative scheme :

$$
\begin{aligned}
{}^{(n+1)}\vec{\psi}^{in} &= \Theta(k)\,{}^{(n)}\vec{\psi}^{in}, \\
\alpha^{(n+1)} &= \left\{ \frac{< {}^{(n+1)}\vec{\psi}^{in}, {}^{(n+1)}\vec{\psi}^{in} >}{< {}^{(n)}\vec{\psi}^{in}, {}^{(n)}\vec{\psi}^{in} >} \right\}^{\frac{1}{2}} .
\end{aligned}
\tag{31}
$$

Here, $< \ , \ >$ denotes the scalar product. When convergence is attained, we must inspect $\alpha(k)$. If $\alpha(k) = \lim_{n\to\infty} \alpha^{(n)}$ thus found is $> 1(< 1)$ the parameter k must be increased(decreased). Newton's chord method is used in order to allow α to go to unity. To advance the speed of k convergence, we used bisection method.

IV. Numerical Tests and Results

To test our method, two benchmark problems of two-energy group model are considered. ONEDANT set down as reference solution. And all problems were calculated for various mesh size. The first test problem is a heterogeneous slab of $20cm$ thickness with linearly isotropic fission source. This problem consists of two regions. The left region (fuel), $10cm$ thick, has $\sigma_1 = 0.3$, $\sigma_2 = 1.0$, $\sigma_{0,1\to1} = 0.27$, $\sigma_{0,1\to2} = 0.01$, $\sigma_{0,2\to1} = 0.001$, $\sigma_{0,2\to2} = 0.9$. This has a fission source of $\nu\sigma_{f1} = 0.0095$, and $\nu\sigma_{f2} = 0.165$. The right region (water), $10cm$ thick, has $\sigma_1 = 0.401$, $\sigma_2 = 1.3$, $\sigma_{0,1\to1} = 0.32$, $\sigma_{0,1\to2} = 0.08$, $\sigma_{0,2\to1} = 0.002$, $\sigma_{0,2\to2} = 1.29$. To solve this problem, the S_4 Gauss-Legendre quadrature set with a pointwise convergence criterion 10^{-6} is used. It must be noted that our method only one mesh to find all information for this problem. For comparison, ONEDANT with a sufficiently fine mesh division (100 meshes) is applied to this problem. Figure 2 show scalar flux as the slab distance changes. In Table 1, k values and computation times are compared. The numerical tests show almost the same about k. And the computation times now refer to a Pentium III 600 MHz PC.

Second problem is a heterogeneous slab of $100cm$ thickness with linearly anisotropic fission source. As in Figure 3, this problem consists of three regions.

Table 1: Speedup results at each problem

Pro.No	IMGF	B.M[a]	ONEDANT
1	18047[b]	74	20
	23.57[c]	0.1	0.01
	0.819595[d]	0.819595	0.819599
	0.000488[e]	0.000488	
2	1463	44	36
	1.86	0.06	0.08
	0.98537	0.98537	0.984852
	0.0526	0.0526	

a: Bisection Method, b: number of iteration, c: computation time d: k_{eff}, e: error(%)

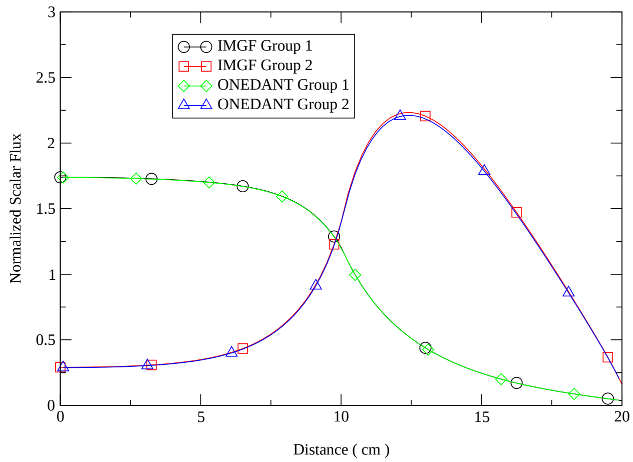


Figure 2: IMGF vs ONEDANT scalar flux of problem 1

The leftmost region (fuel), $10cm$ thick, has $\sigma_1 = 0.3$, $\sigma_2 = 1.0$, $\sigma_{0,1\to1} = 0.27$, $\sigma_{0,1\to2} = 0.01$, $\sigma_{0,2\to1} = 0.001$, $\sigma_{0,2\to2} = 0.9$, $\sigma_{1,1\to1} = 0.09$, $\sigma_{1,1\to2} = 0.002$, $\sigma_{1,2\to1} = 0.0002$, $\sigma_{1,2\to2} = 0.08$. This has a fission source of $\nu\sigma_{f1} = 0.035$, and $\nu\sigma_{f2} = 0.1$. The center region (absorber), $10cm$ thick, has $\sigma_1 = 0.2$, $\sigma_2 = 3.53$, $\sigma_{0,1\to1} = 0.18$, $\sigma_{0,1\to2} = 0.01$, $\sigma_{0,2\to1} = 0.001$, $\sigma_{0,2\to2} = 0.53$, $\sigma_{1,1\to1} = 0.08$, $\sigma_{1,1\to2} = 0.003$, $\sigma_{1,2\to1} = 0.0003$, $\sigma_{1,2\to2} = 0.06$. The right region (water), $80cm$ thick, has $\sigma_1 = 0.401$, $\sigma_2 = 1.30$, $\sigma_{0,1\to1} = 0.32$, $\sigma_{0,1\to2} = 0.08$, $\sigma_{0,2\to1} = 0.002$, $\sigma_{0,2\to2} = 1.29$, $\sigma_{1,1\to1} = 0.07$, $\sigma_{1,1\to2} = 0.003$, $\sigma_{1,2\to1} = 0.0004$, $\sigma_{1,2\to2} = 0.2$. Our result were obtained with the S_4, Gauss-Legendre quadrature sets and only three meshes corresponding to the three regions. For comparison, ONEDANT with a sufficiently fine mesh division (400 meshes) is applied to this problem. Distribution of the scalar flux for this problem is shown in Figure 4.

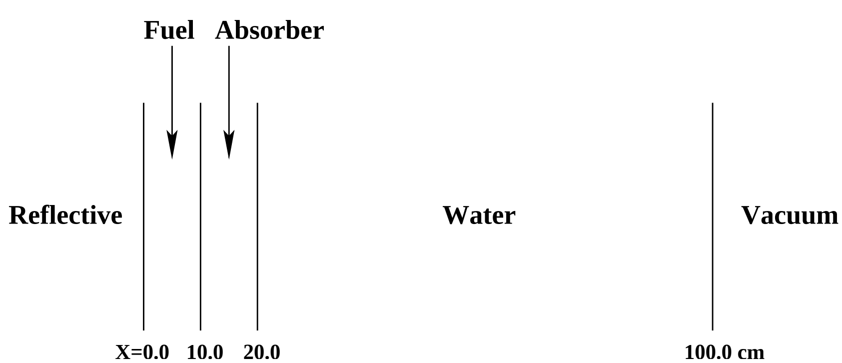
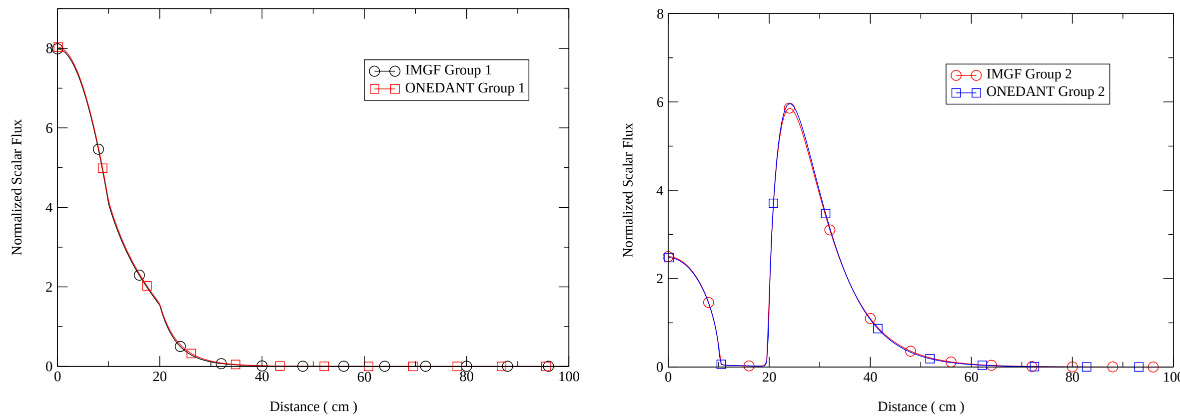


Figure 3: Configuration of Problem 2

V. Conclusions

Analytic solutions of the multigroup discrete ordinates transport equation with linearly anisotropic fission scattering in slab geometry were obtained by using the infinite medium Green's function(IMGF) and Placzek's lemma. The infinite medium Green's function was



(a) Group 1 (b) Group 2

Figure 4: Scalar flux of problem 2

analytically derived by using spectral analysis of the multigroup discrete ordinates transport equation and its transposed equation. This approach leads to an exact relation in which the outgoing angular fluxes are represented in terms of the incoming angular fluxes and interior source. In multi-slab problems, the slabs are coupled through the interface angular fluxes. After the interface angular fluxes are calculated, the analytic solution for each slab is calculated straightforwardly with IMGF. Therefore, in our method, it is sufficient that the necessary number of meshes equal the number of the homogeneous slabs. In eigenvalue problems having fission source, purely complex eigenvalues can occur. So we separated real from purely complex eigenvalues. Then, we made $GN \times GN$ matrices. In this study, we calculated multiplication factor k and scalar flux by using IMGF. For numerical results, we tested two benchmark problems. The numerical tests show that our method gives exact analytic solution of the multigroup discrete ordinates transport equation in slab geometry. Numerical tests show that the IMGF and ONEDANT provide basically the same scalar flux and multiplication factor.

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