Generalization of the Fourier Convergence Analysis in the Neutron Diffusion Eigenvalue Problem

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1. Introduction

Fourier error analysis has been a standard technique for the stability and convergence analysis of linear and nonlinear iterative methods. Lee et al proposed new 2-D/1-D coupling methods and demonstrated several advantages of the new methods by performing a Fourier convergence analysis of the methods as well as two existing methods for a fixed source problem[1].

We demonstrated the Fourier convergence analysis of one of the 2-D/1-D coupling methods applied to a neutron diffusion eigenvalue problem[2]. However, the technique in Ref. 2 cannot be used directly to analyze the convergence of the other 2-D/1-D coupling methods since some algorithm-specific features were used in our previous study.

In this paper we generalized the Fourier convergence analysis technique proposed in Ref. 2 and analyzed the convergence of the 2-D/1-D coupling methods applied to a neutron diffusion eigenvalue problem using the generalized technique.

2. Methods and Results

The 2-D/1-D coupling methods described in Ref. 1 can be directly applied to eigenvalue problems. They begin with the axially averaged 2-D diffusion equation which can be written for each plane as in Eq. (1),

$$-\left(\frac{\partial}{\partial x}D_k\frac{\partial}{\partial x}+\frac{\partial}{\partial y}D_k\frac{\partial}{\partial y}\right)\bar{\phi}_k+\Sigma_k\bar{\phi}_k = \nu\Sigma_{f,k}\bar{\phi}_k/k_{eff}-\left(J_{z,k+1}-J_{z,k}\right)/h_{z,k} \quad (1)$$

and the radially averaged 1-D diffusion equation for each axial mesh which can be written as in Eq. (2),

$$-\frac{\partial}{\partial z}D_{(i,j)}\frac{\partial}{\partial z}\phi_{(i,j)}+\Sigma_{(i,j)}\phi_{(i,j)}=\nu\Sigma_{f,(i,j)}\phi_{(i,j)}/k_{eff} -\left(J_{x,i+1}-J_{x,i}\right)/h_x-\left(J_{y,j+1}-J_{y,j}\right)/h_y \quad (2)$$

Method A in Ref. 1 is to evaluate the TL of the 2-D/1-D equations directly from the 1-D/2-D solutions. The effective multiplication factor can be updated by applying the power iteration as follow :

$$k_{eff}^{(n)}=k_{eff}^{(n-1)}\left(\int_V w\nu\Sigma_f\phi^{(n)}dV\Big/\int_V w\nu\Sigma_f\phi^{(n-1)}dV\right), \quad (3)$$

where w is an arbitrary weighting function.

The model problem in Ref. 2 was used to analyze the convergence of method A applied to an eigenvalue problem. The model problem is a 3-D one-group diffusion eigenvalue problem in a homogeneous finite multiplying medium of N planes with periodic boundary conditions. It is obvious that the exact solution to the model problem is $\phi=\phi_0$ and $k_{eff}=k_\infty=\nu\Sigma_f/\Sigma$. Two basic assumptions are introduced in order to simplify the convergence analysis. These are (1) solving the 2-D problems plane by plane, which means solving them iteratively in the z-direction and (2) solving the 2-D problem by a direct inversion of the 2-D operator in a given plane. The second assumption leads to a zero radial leakage during the iterations, and simplifies Eqs (1) and (2) to:

$$\Sigma\bar{\phi}_k=\nu\Sigma\bar{\phi}_k/k_{eff}-\left(J_{z,k+1}-J_{z,k}\right)/h, \quad (4a)$$

$$-\frac{\partial}{\partial z}D\frac{\partial}{\partial z}\phi_{(i,j)}+\Sigma\phi_{(i,j)}=\frac{1}{k_{eff}}\nu\Sigma\phi_{(i,j)}. \quad (4b)$$

The iterative algorithm of method A applied to the eigenvalue problem with one inner iteration per outer iteration can be expressed by the following equations :

$$\Sigma\bar{\phi}_k^{(n)}=\frac{1}{k_{eff}^{(n-1)}}\nu\Sigma_f\bar{\phi}_k^{(n-1)}-\frac{1}{h}\left(J_{z,k+1}^{(n-1)}-J_{z,k}^{(n-1)}\right), \quad (5a)$$

$$k_{eff}^{(n)}=k_{eff}^{(n-1)}\sum_{k'}\bar{\phi}_{k'}^{(n)}\Big/\sum_{k'}\bar{\phi}_{k'}^{(n-1)}, \quad (5b)$$

$$J_{z,k}^{(n)}=-A^{(n)}\left(\bar{\phi}_k^{(n)}-\bar{\phi}_{k-1}^{(n)}\right), \quad (5c)$$

where

$$A^{(n)}=\frac{D\left(\kappa^{(n)}\right)^2h}{4\sinh^2\left[\kappa^{(n)}h/2\right]};\quad \kappa^{(n)}=\sqrt{\frac{\Sigma-\nu\Sigma_f/k_{eff}^{(n)}}{D}}$$

Note that the two-node analytic nodal method was used to solve the axial 1-D equation and also that a constant weighting function was used to get Eq. (5b).

As we did in the fixed source problem, let's introduce a first order perturbation of $\bar{\phi}_k^{(n)}$, $k_{eff}^{(n)}$, and $A^{(n)}$ in Eq. (5).

$$\bar{\phi}_k^{(n)}=\phi_0\left(1+\varepsilon\xi_k^{(n)}\right), \quad (6a)$$

$$1/k_{eff}^{(n)}=\left(1/k_\infty\right)\left(1+\varepsilon\delta^{(n)}\right), \quad (6b)$$

$$A^{(n)}=A_0\left(1+\varepsilon\theta^{(n)}\right), \quad (6c)$$

where $A_0=\lim_{k_{eff}^{(n)}\to k_\infty}A^{(n)}=D/h$.

Note that $\delta^{(n)}$ and $\theta^{(n)}$ as well as $k_{eff}^{(n)}$ and $A^{(n)}$ are independent of the mesh index k.

Inserting Eq. (6) into Eq. (5) and dropping the $O\left(\varepsilon^2\right)$ terms yields the following linearized equation :

$$\xi_k^{(n)}=\delta^{(n-1)}+\xi_k^{(n-1)}+L^2/h^2\left(\xi_{k-1}^{(n-1)}-2\xi_k^{(n-1)}+\xi_{k+1}^{(n-1)}\right), \quad (7a)$$

$$\delta^{(n)} + \frac{1}{N}\sum_{k'=0}^{N-1}\xi_{k'}^{(n)} = \delta^{(n-1)} + \frac{1}{N}\sum_{k'=0}^{N-1}\xi_{k'}^{(n-1)} . \quad (7b)$$

Note that $\theta^{(n)}$ disappeared in the linearized equation.

There are only N independent bases for the flux vector because the dimension of the flux vector is N. We can choose the N eigenvectors from the lowest mode as the basis. The flux can be expanded by the N eigenvectors, $e^{i\lambda_m x}$ $(m = 0,1,\Lambda\ ,N-1)$, corresponding to the eigenmodes $\lambda_m = 2m\pi/(Nh)$ which satisfy the periodic boundary conditions of the model problem. Therefore, we can introduce the following Fourier ansatz :

$$\xi_k^{(n)} = \xi_{k,m}^{(n)} = a_m \omega_m^n e^{i\lambda_m (k+1/2)h} , \quad (8a)$$

$$\delta^{(n)} = b\omega_0^n , \quad (8b)$$

Note that only some discrete values of the wave number, λ_m, are allowed in Eq. (8) whereas a continuous wave number is allowed in the fixed source problem. Among the eigenmodes, $\lambda_0 = 0$ forms the fundamental mode solution of the flux, $1+\xi_{k,0}^{(n)}$ for the model problem, and the other modes, $\lambda_m (m = 1,2,\Lambda\ ,N-1)$, form the higher mode error term of the flux, $\xi_{k,m}^{(n)}$. As indicated above, $\delta^{(n)}$ is independent of the space, which means that only $\lambda_0 = 0$ is allowed for the wave number of the Fourier ansatz of $\delta^{(n)}$.

It is trivial to show that

$$\sum_{k'=0}^{N-1}\xi_{k',m}^{(n)} = 0 \ \ (m = 1,2,\Lambda\ ,N-1) . \quad (9)$$

Using Eq. (9), we can simply Eq. (7) for $m > 0$:

$$\xi_{k,m}^{(n)} = \xi_{k,m}^{(n-1)} + \left(L^2/h^2\right)\left(\xi_{k-1,m}^{(n-1)} - 2\xi_{k,m}^{(n-1)} + \xi_{k+1,m}^{(n-1)}\right) . \quad (10)$$

We can also simplify Eq. (7) for $m = 0$:

$$\xi_{*,0}^{(n)} = \delta^{(n-1)} + \xi_{*,0}^{(n-1)} , \quad (11a)$$

$$\delta^{(n)} + \xi_{*,0}^{(n)} = \delta^{(n-1)} + \xi_{*,0}^{(n-1)} , \quad (11b)$$

where $\xi_{*,0}^{(n)} = \xi_{k,0}^{(n)} = a_0\omega_0^n$.

From Eq. (10) and (11), we get

$$\omega_0 = 0 , \quad (12a)$$

$$\omega_m = 1 + 2\left(L^2/h^2\right)\left[\cos\left(\tau_m\right) - 1\right] \quad (m > 0) . \quad (12b)$$

Note that $\tau_m = \lambda_m h$ are $2\pi/N$, $4\pi/N$, Λ , $2(N-1)\pi/N$.

The spectral radius of the linearized algorithm of method A for the eigenvalue problem is given by :

$$\rho = \underset{m=0,1,\Lambda\ ,N-1}{\text{Max}} \left|\omega_m\right| . \quad (13)$$

One can directly apply the Fourier convergence analysis demonstrated above to the other 2-D/1-D coupling methods. Figure 1 shows the spectral radius of the 2-D/1-D coupling methods as a function of the axial mesh size for the model problem with $N = 4$, $D = 0.833333$, $\Sigma = 0.02$, and $\nu\Sigma_f = 0.019$. The line is the analytic spectral radius obtained by the Fourier analysis and the dots are the numerical ones. The large gray dots are used for method D to distinguish them from those for method C. As indicated, a good agreement is observed between the analytic and numerical results. As we expected, we got the same result as that in Ref. 2 for method A. Though method A is the best in terms of a spectral radius for a large mesh size, it diverges for a small mesh size. The other methods are always stable regardless of the axial mesh size. The spectral radius of methods C and D are smaller than that of method B in the range of a practical mesh size. The spectral radius of methods C and D are identical while that of D is smaller than that of C in a fixed source problem[1]. It is interesting that the spectral radius in the eigenvalue problem approaches 1 as the mesh size increases while it approaches zero in the fixed source problem.

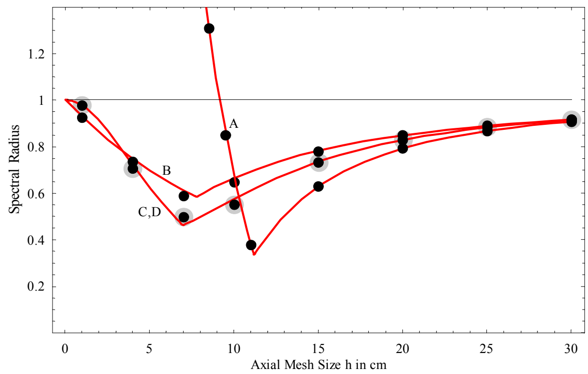

Figure 1. The spectral radius of 2-D/1-D coupling methods

3. Conclusion

In this paper we generalized the Fourier convergence analysis technique proposed in Ref. 2 and analyzed the convergence of the 2-D/1-D coupling methods applied to a neutron diffusion eigenvalue problem using the generalized technique. The analysis showed that the newly proposed methods, C and D, are better than the existing methods, A and B, even for the eigenvalue problems.

REFERENCES

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