# Theoretical Approach to Transport Acceleration using an Operator Form of Synthetic Equation

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## 1. Introduction

The discrete ordinates form ($S_N$) of neutron transport equation becomes more practical with the use of appropriate acceleration schemes. The well-known acceleration scheme for this purpose is a diffusion synthetic acceleration (DSA). The acceleration consists of solving alternate transport and low-order acceleration equation. In this work the derivations of low-order acceleration equations are generalized in an operator form of synthetic acceleration. The source iteration, $P_0$, $P_1$ (equivalent to DSA) and $P_2$ acceleration schemes for one-dimensional slab geometry are derived and the convergence rate of each scheme is compared by performing theoretical Fourier analyses.

## 2. Methods and Results

### *2.1 Operator Form of Synthetic Acceleration*

The transport equation may be written in an operator form as

$$H\chi = Q, \tag{1}$$

where $H$ is the even-parity transport operator

$$H = -\mu^2 \frac{d}{dx}\frac{1}{\sigma}\frac{d}{dx} + \sigma - \sigma_s \int_0^1 d\mu, \tag{2}$$

and $\chi$ is the even-parity angular flux. Here, we use the second-order even-parity equation as the transport equation. The choice of the first-order or second-order form of transport equation does not change our development, since they are algebraically same equations. Practically Eq.(1) is solved iteratively as

$$H_0 \chi^{l+1/2} = H_1 \phi_0^l + Q, \tag{3}$$

with the iteration index $l$. Here, $H_0$ is the streaming-collision operator and $H_1$ is the isotropic scattering operator. To derive the acceleration method we integrate Eq.(1) over $\mu$ and write the operator $H$ using $\mu$-independent operator $H_L$, and a correction $H - H_L$ as

$$\int_0^1 [H_L + (H - H_L)]\chi d\mu = Q. \tag{4}$$

Since $H_L$ is independent of $\mu$, we may write

$$\int_0^1 H_L \chi d\mu = H_L \phi_0. \tag{5}$$

Then rearranging terms in Eq.(4) yields

$$H_L \phi_0 = Q - \int_0^1 (H - H_L)\chi d\mu. \tag{6}$$

This expression suggests the operator form of synthetic acceleration:

$$H_L \phi_0^{l+1} = Q - \int_0^1 (H - H_L)\chi^{l+1/2} d\mu, \tag{7}$$

where $\chi^{l+1/2}$ is determined from Eq.(3). Now the acceleration consists of solving alternate transport [Eq.(3)] and low-order equation [Eq.(7)].

### *2.2 Low-Order Equations*

In this paper our goal is to determine the optimum value of $N$ in $P_N$ acceleration and we analyze the spectral radius of iteration process by performing a Fourier analysis for an infinite medium problem. However, we have little incentive to consider a method for $N$>2. The reason is that excessive run time would be required to solve the acceleration equation in the case of $N$>2. To derive $P_N$ ($N$=0,1,2) accelerating equation, we take the $P_N$ operator as a low-order operator $H_L$ and the even-parity transport operator as a high-order operator $H$ in Eq.(7) . The results for $P_0$, $P_1$, and $P_2$ accelerations are as follows;

$$\sigma_a \phi_0^{l+1} = Q + \frac{2}{3}\frac{d}{dx}\frac{1}{\sigma}\frac{d\phi_2^{l+1/2}}{dx} + \frac{1}{3}\frac{d}{dx}\frac{1}{\sigma}\frac{d\phi_0^{l+1/2}}{dx}, \tag{8}$$

$$-\frac{d}{dx}\frac{1}{3\sigma}\frac{d\phi_0^{l+1}}{dx} + \sigma_a \phi_0^{l+1} = Q + \frac{2}{3}\frac{d}{dx}\frac{1}{\sigma}\frac{d\phi_2^{l+1/2}}{dx}, \tag{9}$$

$$-\frac{d}{dx}\frac{1}{3\sigma}\frac{d}{dx}\left(1 + \frac{4\sigma_a}{5\sigma}\right)\phi_0^{l+1} + \sigma_a \phi_0^{l+1}$$
$$= Q + \frac{2}{3}\frac{d}{dx}\frac{1}{\sigma}\frac{d\phi_2^{l+1/2}}{dx} - \frac{4}{15}\frac{d}{dx}\frac{1}{\sigma}\frac{d}{dx}\left(\frac{\sigma_a}{\sigma}\right)\phi_0^{l+1/2}, \tag{10}$$

where $\phi_2$ is the second Legendre moment.

### *2.2 Fourier Analysis and Result*

Acceleration efficiency for the transport iterations can be analytically specified by the Fourier analysis. The error form of the transport equation, with the source term vanished, can be recast using new variables

$$x = \int_0^x \sigma(x')dx',\ c = \sigma_s/\sigma,\ S = Q/\sigma,$$

then the angular and scalar fluxes between the adjacent iterations can be expanded by the infinite Fourier complex series as

$$\tilde{\chi}^{l+1/2} = \sum_{k=-\infty}^{\infty} \omega_k^l(\lambda_k) a_k(\lambda_k, \mu) e^{i\lambda_k x}, \tag{11}$$

$$\tilde{\phi}_0^l = \sum_{k=-\infty}^{\infty} \omega_k^l(\lambda_k) e^{i\lambda_k x}. \tag{12}$$

Here, $\omega_k$ is the eigenvalue of the $k$'th harmonic and it is a function of the parameter $\lambda_k$. Then, the propagation of error is obviously governed by the types of equations and the convergence is studied by noting the behavior of $\omega_k$. The Fourier analyses yield the eigenvalues for $P_0$, $P_1$, and $P_2$ accelerations, respectively, as

$$\omega_{P_0} = \frac{c\left(\tan^{-1}\lambda/\lambda-1\right)}{1-c}, \tag{13}$$

$$\omega_{P_1} = \frac{c\left[\left(\lambda/3+1/\lambda\right)\tan^{-1}\lambda-1\right]}{\lambda^2/3+\left(1-c\right)}, \tag{14}$$

$$\omega_{P_2} = \frac{c\left[\left\{\left(9-4c\right)\lambda/15+1/\lambda\right\}\tan^{-1}\lambda-1\right]}{\left(9-4c\right)\lambda^2/15+\left(1-c\right)}. \tag{15}$$

In Figure 1 and 2, the $\omega$'s of the source iteration, $P_0$, $P_1$, and the $P_2$ accelerations are plotted against the parameter λ for $c$=1 and $c$=0.5, respectively. When the scattering ratio is unity ($c$=1), the $P_2$ acceleration behaves exactly same as the $P_1$ acceleration, however, the $\omega$ of $P_1$ acceleration is always smaller than that of the $P_2$ acceleration for 0<$c$<1. It means that the $P_1$ acceleration generally converges faster than the $P_2$ acceleration. To examine the convergence rate, we usually define the spectral radius

$$\rho \equiv \sup_{\lambda} |\omega| = \sup_{\lambda}\left|\frac{\varphi_0^l-\varphi_0^{l-1}}{\varphi_0^{l-1}-\varphi_0^{l-2}}\right| \tag{16}$$

and it measures the slowest possible reduction of error in scalar flux from one iteration to the next iteration. The spectral radii of the source iteration, $P_0$, $P_1$, and $P_2$ accelerations are given in Table 1.

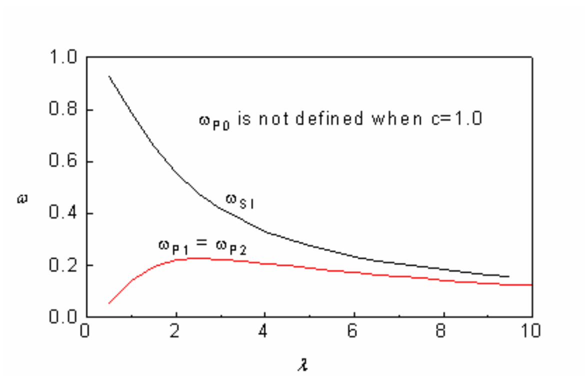


Figure 1  ω of  Acceleration ($c$=1.0)

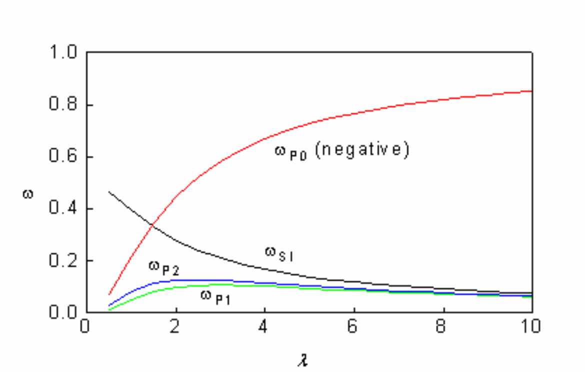


Figure 2  ω of  Acceleration ($c$=0.5)

Table 1.  Spectral Radius ρ of Acceleration

| Scheme | λ for max(ω) | ρ |
|---|---|---|
| Source Iteration | 0.0 | 1.0 |
| $P_0$ acceleration | ∞ | 1.0 |
| $P_1$ acceleration | 2.5 | 0.2247 |
| $P_2$ acceleration | 2.5 | 0.2247 |

## 3. Conclusion

The task of the low-order equation is to solve a problem for the iteration error. Therefore, we no longer tell the difference in accuracy of accelerating equations no matter what order of $P_N$ approximations are used. Any difference is only in acceleration efficiency because of differing spectral radii of the iteration algorithms. Thus, one cannot simply assume that higher accuracy in a low-order operator will automatically lead to better acceleration performance. By the Fourier analysis, we conclude that the $P_1$ acceleration has the smallest spectral radius among the source iteration, $P_0$, and the $P_1$ acceleration schemes.

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