Analytical Derivation of the Compton Wavelength Shift

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1. Introduction

In quantum mechanics, the Compton scattering is the increase in wavelength which means decrease in energy. It occurs when X-ray photon with energy around 0.5MeV to 3.5MeV with an electron in a material[1]. The amount of the wavelength increase is so-called the Compton wavelength shift. For freshmen of the nuclear physics course, the Compton shift was derived by the two types of the analytical methods.

2. Methods and Result

In the Compton scattering, a X-ray photon of wavelength λ comes in an atom, collides with a stationary electron, and a new X-ray photon of wavelength λ' is scattered at an angle Θ and a recoil electron e' is ejected. From the energy and momentum conservation, Eq.(1) and Eq.(2) can be given as follows:

$$E_X + E_e = E_{X'} + E_{e'} \quad (1)$$

$$\mathbf{p}_X = \mathbf{p}_{X'} + \mathbf{p}_{e'},\ \mathbf{p}_e = 0 \quad (2)$$

From Eq.(1) for the energy part, Eq.(3) can be rearranged as follow[2][3]:

$$hc/\lambda + m_e c^2 = hc/\lambda' + [(p_{e'}c)^2 + (m_e c^2)^2]^{1/2} \quad (3)$$

Solving Eq.(3) for $p_{e'}^2$, Eq.(4) can be given.

$$p_{e'}^2 = \{[hc/\lambda + m_e c^2 - hc/\lambda']^2 - m_e^2 c^4\}/c^2 \quad (4)$$

By rearranging Eq.(2), $\mathbf{p}_{e'} = \mathbf{p}_X - \mathbf{p}_{X'}$ and also by squaring its both side,

$$\begin{aligned} \mathbf{p}_{e'}^2 &= (\mathbf{p}_X - \mathbf{p}_{X'}) \cdot (\mathbf{p}_X - \mathbf{p}_{X'}) \\ &= \mathbf{p}_X^2 + \mathbf{p}_{X'}^2 - 2\mathbf{p}_X \cdot \mathbf{p}_{X'} \\ &= \mathbf{p}_X^2 + \mathbf{p}_{X'}^2 - 2\,|\mathbf{p}_X| \cdot |\mathbf{p}_{X'}| \cos\Theta \end{aligned} \quad (5)$$

Accordingly, Eq.(5) can be given.

$$p_{e'}^2 = (h/\lambda)^2 + (h/\lambda')^2 - 2h^2\cos\Theta/\lambda\lambda' \quad (6)$$

Next, by equalizing the right-hand sides of Eq.(4) and Eq.(5), and rearranging Eq.(6) can be obtained.

$$hc^2\cos\Theta/\lambda\lambda' = hc^2/\lambda\lambda' - (c/\lambda - c/\lambda')m_e c^2 \quad (7)$$

Now, by solving Eq.(6) into factors by $hc^2/\lambda\lambda'$ and rearranging, Eq.(7) can be obtained.

$$(c/\lambda - c/\lambda')m_e c^2 = hc^2(1 - \cos\Theta)/\lambda\lambda' \quad (8)$$

By multiplying Eq.(7) by $\lambda\lambda'$ and rearranging, the following Eq.(9) can be derived.

$$\lambda' - \lambda = (h/m_e c)(1 - \cos\Theta) \quad (9)$$

where, $h/m_e c$ is called Compton wavelength (λ_c). Its value is $\lambda_c = 2.426 \times 10^{-3}$ nm.

From now, the Compton wavelength shift is derived using the two dimensional method for convenience. That is, in case of the collision of X-ray photon with an electron at rest, the energy conservation is effected between them.

Before the Compton scattering, the energy of incident X-ray photon is hc/λ while the kinetic energy of an electron is zero, After the scattering, however, the energy of a scattered X-ray photon is hc/λ' and the kinetic energy of a scattered electron is $\frac{1}{2}m_e v^2$. Accordingly,

$$hc/\lambda = hc/\lambda' + \tfrac{1}{2}m_e v^2 \qquad (10)$$

Meanwhile, the conservation of momentum is also applied to such a collision described above. Before and after Compton scattering, Eqs,(11) and (12) can be given as the X-axis component and Y-axis component, considering the recoiling angle of electron ϕ, as follows:

X-component: $h/\lambda = h\cos\Theta/\lambda' + m_e v\cos\phi \qquad (11)$

Y-component: $0 = -h\sin\Theta/\lambda' + m_e v\sin\phi \qquad (12)$

Accordingly, the following relatioships are made.

From Eq.(11), $m_e v\cos\phi = h/\lambda - h\cos\Theta/\lambda' \qquad (13)$

From Eq.(12), $m_e v\sin\phi = h\sin\Theta/\lambda' \qquad (14)$

By squaring both sides of Eqs.(13) and (14), and combining them, Eq.(15) is obtained by introducing the relatioship $\sin^2\phi + \cos^2\phi = 1$.

$$m_e^2 v^2 = h^2/\lambda^2 + h^2/\lambda'^2 - 2h\cos\Theta/\lambda\lambda' \qquad (15)$$

In addition, rearranging Eq.(10) for v^2, the following Eq.(16) is obtained.

$$v^2 = 2h\,(c/\lambda - c/\lambda')/m_e \qquad (16)$$

Now, by substituting Eq.(16) to Eq.(15) and rearranging, the following Eq.(17) is obtained.

$$2m_e hc(\lambda'-\lambda)/\lambda\lambda' = h^2/\lambda^2 + h^2/\lambda'^2 - 2h^2\cos\Theta/\lambda\lambda' \qquad (17)$$

Where, by assuming $\lambda^2 \fallingdotseq \lambda'^2 \fallingdotseq \lambda\lambda'$ and rearranging Eq.(17), the following Eq.(18) can also be derived as in the result of Eq.(9).

$$\lambda' - \lambda \fallingdotseq (h/m_e c)(1 - \cos\Theta) \qquad (18)$$

In Eq.(9) and Eq.(18) above, therefore, the Compton wavelength shift depends upon only the scattering angle(Θ) of the X-ray photon.

3. Conclusions

In nuclear physics, Compton effect means the increase in wavelength of X-ray photon. In Eqs.(9) and (18), the Compton wavelength shift depends only on the scattering angle of an X-ray photon Θ which is the angular deviation from its original path. From this point of view, the Compton wavelength shift $\triangle\lambda=\lambda'-\lambda$ were analytically derived for the beginners of nuclear physics course.

NOMENCLATURES

λ : wavelength of incident X-ray photon
λ': wavelength of scatterd X-ray photon
h : Planck constant
m_e : the mass of an electron at rest
c : the speed of light

REFERENCES

[1] http://www.experiencefestival.com/a/Compton_scattering/id/2000548
[2] Marcelo Alonso, Edward J. Finn, PHYSICS, Wesley Publishing Co., Inc., Reading, MA, pp.587-589, 1975.
[3] J. Kenneth Shultis and Richard E. Faw, Radiation Shielding, American Nuclear Society, Inc., Lagrange Park, IL., pp.31-34, 2000.