A New SP_n Theory Formulation with Self-consistent Physical Assumptions on Angular Flux

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1. Introduction

The SP_n equations were initially proposed by Gelbard without adequate mathematical derivation or justification [1]. The 1D P_n equations are generalized to 3D SP_n equations by directly replacing its 1D gradient operator with 3D gradient operator. Later on more sophisticated mathematical formulations using either the asymptotic approximation or the variational method were introduced for better mathematical justification, in particular for the SP_3 case [2, 3]. Although the SP_3 equations can be derived in these formulations, they do not provide a way to explicitly reconstruct the angular flux representation from the SP_3 solution. One does not have a “physical picture” for the angular flux for the SP_n solution. It is therefore not possible to compare the angular distribution of the correct transport solution to the approximate SP_n solution, nor to extract from a reference transport solution the corresponding SP_n solution. This makes it difficult to understand and visualize the physical meaning of the SP_n approximation. Moreover this causes a problem in practical engineering applications of SP_n, as one cannot introduce correction factors, such as the popular discontinuity factors, to compensate for the approximations in the SP_n solution. It has been demonstrated that SP_3 calculation per se without using any correction factor cannot compete in accuracy with the conventional diffusion calculation using discontinuity factors [4].

To resolve the above cited problem Chao and Yamamoto [5] adopted the physical interpretation of the SP_n model as the neutron transport being “locally 1D” at any point in space [2]. They pointed out that this necessarily implies the angular flux distribution being always cylindrically symmetric with respect to the net current whose direction may change continuously thru space. In this picture the angular flux is an expansion in Legendre polynomials of the cosine of the polar angle with respect to the net current direction. Chao and Yamamoto showed how the SP_n equations can be derived by plugging this angular flux function into the transport equation and assuming certain specific approximations. This explicit angular flux representation was then used to calculate the SP_3 discontinuity factors to show that the SP_3 superiority over diffusion can be restored when discontinuity factors are applied to SP_3 calculation as well [4, 6].

However the assumption of local 1D planar behavior of the angular flux is not consistent with the fact that in SP_n equations at any spatial point there may exist many vector directions given by the gradients of the flux moment functions, $\nabla\phi_n(r)$, which can all be different from the direction of the net current *J*. As a result, Chao and Yamamoto had to make additional ad hoc assumptions to derive the SP_n equations and for the generation of the correct expression of the net current in terms of the gradients of the 0^{th} and 2^{nd} order flux moments $\nabla\phi_0(r)$ and $\nabla\phi_2(r)$. In this paper we propose that a self-consistent physical model for SP_n is a combination of multiple locally planar functions along different directions instead of along the net current direction alone. Each order of the flux moments contributes to one of the superposed locally 1D functions. The angular distribution of the n^{th} order flux moment is the n^{th} order Legendre polynomial of the cosine of the polar angle with respect to the direction of the spatial gradient, $\nabla\phi_n(r)$, of the n^{th} order flux moment. With this physical model one can rigorously derive the equations for the current and the boundary conditions. The SP_n equations can also be derived with the additional assumption of the total cross-section being locally nearly flat, which is practically always valid when the spatial variation is discretized in numerical calculations. However the boundary conditions turn out to be different from the conventional ones, containing some non-linear factors involving the cosine of the angle between the boundary surface normal vector and the spatial gradient vectors of the flux moments. The internal interface boundary conditions are not affected by the non-linear factors as they cancel out on the interface. But the external boundary condition does get affected by the non-linear factors. The effect of non-linear factors is of higher order, which disappears if the spatial gradients are parallel to the surface normal vector. So if the non-linear factors are neglected, the external boundary condition also reduces to the conventional one. The non-linear external boundary condition can nevertheless be iteratively updated to estimate the correction effect.

2. The theory and the equation derivation

2.1 The physical model for angular flux representation

For simplicity we will consider the case of mono-energetic isotropic scattering, with the neutron transport equation given in terms of the even and odd parity angular fluxes as,

$$\Omega \bullet \nabla[\psi_O(\Omega,r)] + \Sigma_t \psi_E(\Omega,r) = \frac{Q(r)}{4\pi} \quad (1)$$

$$\psi_O(\Omega,r) = -\frac{1}{\Sigma_t}\Omega \bullet \nabla \psi_E(\Omega,r) \quad (2)$$

In the P_n method the angular flux is expanded in orthogonal spherical harmonics. The n^{th} order moment term in the expansion contains (2n+1) components, each of which is a product of an angular part $Y_{n,m}(\Omega)$ and a spatial part $\phi_{n,m}(r)$ with m between $-n$ and n. The fundamental problem in SP_n is that instead of (2n+1) components it has only one component, $\phi_n(r)$, for the nth moment. The question is then how to replace the (2n+1) components with a single "effective" one and how the corresponding angular distribution looks like for the nth flux moment. Noting that the only information provided by the solution of SP_n equations for the nth flux moment are $\phi_n(r)$ and $\nabla\phi_n(r)$, one can argue that in absence of other information a self-consistent physical picture for the angular distribution of the nth flux moment is cylindrically symmetric with respect to $\nabla\phi_n(r)$, which is the only direction vector available for the nth flux moment. Therefore we propose the following explicit angular flux representation for the SP_n theory, where Ω_n is defined as the unit vector along the direction of the gradient $\nabla\phi_n(r)$,

$$\psi_E(\Omega,r)=\sum_{n=even}\frac{2n+1}{4\pi}P_n(\Omega\bullet\Omega_n)\phi_n(r) \tag{3}$$

$$\psi_O(\Omega,r)=-\frac{1}{\Sigma_t}\sum_{n=even}\frac{2n+1}{4\pi}[P_n(\Omega\bullet\Omega_n)\Omega\bullet\Omega_n][\Omega_n\bullet\nabla\phi_n(r)] \tag{4}$$

Eq. (4) is obtained by plugging Eq. (3) into Eq. (2).

It must be emphasized that there is a catch here. Although the magnitude of the unit vector Ω_n does not change, its direction does vary in space. Therefore when deriving Eq. (4), the spatial gradient operator in Eq. (2) ought to apply to the argument of the Legendre polynomials as well, which has been ignored. The implied assumption is that the rate of the directional change of the gradient is much smaller than the rate of the change of the magnitude itself. This defines what it means that the function in question has a 1D behavior locally. From here on we will regard the unit vector Ω_n as a locally constant vector by ignoring its spatial gradient. With this assumption alone, the equations for the net current and the boundary conditions will be rigorously derived respectively in Sec. 2.2 and Sec. 2.4. The SP_n equations will be derived in Sec. 2.3 with the additional assumption of the total cross-section being locally nearly flat.

2.2 The net current equation

To derive the equation for the net current, we multiply the odd parity angular flux of Eq. (4) with the solid angle vector Ω to integrate over the whole angular space,

$$J=-\frac{1}{\Sigma_t}\sum_{k=even}\frac{2k+1}{4\pi}\int\Omega P_k(\Omega\bullet\Omega_k)\Omega\bullet\Omega_k[\Omega_k\bullet\nabla\phi_k(r)]d\Omega \tag{5}$$

The even parity angular flux will not contribute because of its symmetry in Ω. Each integral in Eq. (5) can be evaluated independently, where we choose Ω_k as the z-axis. After integrating over the azimuthal angle, Eq. (5) becomes,

$$J=-\frac{1}{\Sigma_t}\sum_{k=even}\frac{2k+1}{2}[\int_{-1}^{1}\mu^2P_k(\mu)d\mu]\nabla\phi_k(r) \tag{6}$$

The integral can be easily calculated using the Legendre polynomial recursion relation and orthogonality relation. The result is,

$$J=-\frac{1}{\Sigma_t}\sum_{k=even}[\frac{1}{3}\delta_{0,k}+\frac{(k-1)k}{2k-1}\delta_{0,k-2}]\nabla\phi_k(r) \tag{7}$$
$$=-\frac{1}{3\Sigma_t}[\nabla\phi_0(r)+2\nabla\phi_2(r)]$$

The above result is identical to the familiar one in the conventional SP_3 equation. Here it holds generically for SP_n regardless of the order n to choose. The higher order moments in Eq. (6) drop out after integration.

2.3 The SPn equations

To derive the equations for $\phi_n(r)$, we plug Eq. (3) and Eq. (4) into Eq. (1) and multiply both sides of the equation with $P_n(\Omega\cdot\Omega_n)$ to integrate it over the angular space,

$$-\sum_{k=even}\frac{2k+1}{4\pi}\int\{P_n(\Omega\bullet\Omega_n)P_k(\Omega\bullet\Omega_k)[\Omega\bullet\nabla\frac{1}{\Sigma_t}\Omega\bullet\nabla-\Sigma_t]\phi_k(r)\}d\Omega \tag{8}$$
$$=Q\delta_{n0}$$

Note that the two Legendre polynomials in the integrand have different arguments. Nevertheless the following orthogonality relation still holds, which can be readily proved by using the Addition Theorem for Legendre polynomial [7],

$$\frac{2k+1}{4\pi}\int P_n(\Omega\bullet\Omega_A)P_k(\Omega\bullet\Omega_B)d\Omega=\delta_{nk}P_n(\Omega_A\bullet\Omega_B) \tag{9}$$

Making use of Eq. (9) we get from Eq. (8),

$$-\sum_{k=even}\{\frac{2k+1}{4\pi}\int P_n(\Omega\bullet\Omega_n)P_k(\Omega\bullet\Omega_k)[\Omega\bullet\nabla\frac{1}{\Sigma_t}\Omega\bullet\nabla\phi_k(r)]d\Omega\} \tag{10}$$
$$+\Sigma_t\phi_n(r)=Q\delta_{n0}$$

Recalling that Ω_k is a locally constant vector, we can rewrite the vector product in the integrand of Eq. (10) as,

$$\Omega\bullet\nabla\frac{1}{\Sigma_t}\Omega\bullet\nabla\phi_k(r)=\Omega\bullet\nabla[\frac{1}{\Sigma_t}(\Omega\bullet\Omega_k)(\Omega_k\bullet\nabla\phi_k(r))] \tag{11}$$
$$=(\Omega\bullet\Omega_k)\Omega\bullet\nabla[\Omega_k\bullet\frac{\nabla\phi_k(r)}{\Sigma_t}]$$

It can be proved via vector algebra that the part in Eq. (11) involving the gradient operator can be written as,

$$\nabla[\Omega_k\bullet\frac{1}{\Sigma_t}\nabla\phi_k(r)]=\Omega_k[\nabla\bullet\frac{1}{\Sigma_t}\nabla\phi_k(r)]+\Omega_k\times[(\nabla\frac{1}{\Sigma_t})\times\nabla\phi_k(r)] \tag{12}$$

We will first assume that the second term in Eq. (12) can be neglected with its justification to be discussed later,

$$\nabla[\Omega_k\bullet\frac{1}{\Sigma_t}\nabla\phi_k(r)]\cong\Omega_k[\nabla\bullet\frac{1}{\Sigma_t}\nabla\phi_k(r)] \tag{13}$$

Plugging Eq. (13) and Eq. (11) into Eq. (10) gives,

$$-\sum_{k=even}\{\frac{2k+1}{4\pi}[\int P_n(\Omega\bullet\Omega_n)P_k(\Omega\bullet\Omega_k)(\Omega\bullet\Omega_k)^2d\Omega][\nabla\bullet\frac{1}{\Sigma_t}\nabla\phi_k(r)]\}$$
$$+\Sigma_t\phi_n(r)=Q\delta_{n0} \tag{14}$$

The integral in Eq. (14) can be easily calculated by using the recursion relation, and the orthogonal relation of Eq. (9). The result is exactly the SP_n equations,

$$-\frac{n(n-1)}{(2n+1)(2n-1)}\nabla\bullet\frac{\nabla\phi_{n-2}(r)}{\Sigma_t}-\frac{(2n^2+2n-1)}{(2n+3)(2n-1)}\nabla\bullet\frac{\nabla\phi_n(r)}{\Sigma_t} \tag{15}$$
$$-\frac{(n+1)(n+2)}{(2n+1)(2n+3)}\nabla\bullet\frac{\nabla\phi_{n+2}(r)}{\Sigma_t}+\Sigma_t\phi_n(r)=Q\delta_{n0}$$

The first term in Eq. (12) is a vector in the direction of the gradient of the flux moment, while the second term there is a vector transverse to the direction of the gradient of the flux moment. To be consistent with the physical picture of our proposed model, we would expect the transverse vector term to be small. Had the second term in Eq. (12) been kept, Eq. (15) would have additional terms involving the following magnitude scalar of the vector cross product,

$$\left|(\nabla \frac{1}{\Sigma_t}) \times \nabla \phi_k(r)\right| \tag{16}$$

This magnitude scalar vanishes in two limits: either the gradient of the cross-section being parallel to the gradient of the flux moment or the cross-section being locally almost flat such that its gradient being very small. The former limit would require the gradients of all the flux moments being simultaneously parallel to the gradient of the cross-section. This is the very strong traditional assumption that the problem is locally a planar problem with only one unique gradient direction, such as a truly 1D problem. The latter limit is less stringent and requires only a mild gradient of the cross-section compared to the flux gradient. This requirement is reasonable. In fact this approximation is always made in practice when spatial discretization is used with flat cross-section in discretized meshes.

2.4 Boundary conditions

The boundary conditions are provided by the relation between the incoming and outgoing partial currents. Thus we need to calculate the partial currents in the context of SP_n formulation. We will first calculate the generic n^{th} order moment of the partial currents and then use it to derive the boundary condition equations.

2.4.1 The n^{th} order moment of partial currents

The angular partial current going out or in through a surface with the normal vector $\hat{n}$ is defined as,

$$j^{\pm}_{\hat{n}\bullet\Omega>0}(\Omega, r) = \hat{n}\bullet\Omega[\psi_E(\Omega, r) \pm \psi_O(\Omega, r)] \tag{17}$$

To calculate the n^{th} order moment of the partial currents we multiply Eq. (17) with the n^{th} even order Legendre polynomial of the cosine of the polar angle with respect to the normal vector $\hat{n}$, and then integrate it over the angular space. Introducing J_n for the n^{th} order net current projection in the normal direction, and Φ_n the n^{th} order projection of half of the sum of incoming and outgoing partial currents, Eq. (17) can be written as,

$$J^{\pm}_n = \Phi_n \pm \frac{1}{2} J_n \tag{18}$$

$$J_n = \int P_n(\Omega\bullet\hat{n})\hat{n}\bullet\Omega\psi_O(\Omega, r)d\Omega \tag{19}$$

$$\Phi_n = \frac{J^+_n + J^-_n}{2} = \int_{\hat{n}\bullet\Omega>0} P_n(\Omega\bullet\hat{n})(\hat{n}\bullet\Omega)\psi_E(\Omega, r)d\Omega \tag{20}$$

To calculate J_n we plug Eq. (4) into Eq. (19), and use the recursion relation and Eq.(9). The result is,

$$\begin{aligned} J_n = &-\frac{1}{\Sigma_t}\frac{n(n-1)}{(2n+1)(2n-1)}P_{n-1}(\hat{n}\bullet\Omega_{n-2})\Omega_{n-2}\bullet\nabla\phi_{n-2}(r) \\ &-\frac{1}{\Sigma_t}\frac{1}{2n+1}[\frac{(n+1)^2}{2n+3}P_{n+1}(\hat{n}\bullet\Omega_n) + \frac{n^2}{2n-1}P_{n-1}(\hat{n}\bullet\Omega_n)]\Omega_n\bullet\nabla\phi_n(r) \\ &-\frac{1}{\Sigma_t}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+1}(\hat{n}\bullet\Omega_{n+2})\Omega_{n+2}\bullet\nabla\phi_{n+2}(r) \end{aligned} \tag{21}$$

The above equation can be rewritten in a physically more appealing form,

$$\begin{aligned} J_n = &-\frac{1}{\Sigma_t}\frac{n(n-1)}{(2n+1)(2n-1)}\frac{P_{n-1}(\hat{n}\bullet\Omega_{n-2})}{\hat{n}\bullet\Omega_{n-2}}\hat{n}\bullet\nabla\phi_{n-2}(r) \\ &-\frac{1}{\Sigma_t}\frac{1}{2n+1}[\frac{(n+1)^2}{2n+3}\frac{P_{n+1}(\hat{n}\bullet\Omega_n)}{\hat{n}\bullet\Omega_n} + \frac{n^2}{2n-1}\frac{P_{n-1}(\hat{n}\bullet\Omega_n)}{\hat{n}\bullet\Omega_n}]\hat{n}\bullet\nabla\phi_n(r) \\ &-\frac{1}{\Sigma_t}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}\frac{P_{n+1}(\hat{n}\bullet\Omega_{n+2})}{\hat{n}\bullet\Omega_{n+2}}\hat{n}\bullet\nabla\phi_{n+2}(r) \end{aligned} \tag{22}$$

In Eq. (22) the Legendre polynomial terms all appear as the ratio of an odd order polynomial divided by its argument, which is a symmetric function. And the ratio approaches unity as its argument approaches unity, which happens when the gradient vector is parallel to the surface normal direction vector.

To calculate Φ_n we plug Eq. (3) into Eq. (20). Taking the surface normal vector $\hat{n}$ as the z-axis, we get,

$$\Phi_n = \sum_{k=even}\frac{2k+1}{4\pi}\{\int_0^1 P_n(\cos(\theta))\cos(\theta)[\int_0^{2\pi} P_k(\Omega\bullet\Omega_k)d\varphi]d\cos(\theta)\}\phi_k(r) \tag{23}$$

Using the Addition Theorem for Legendre polynomial [7], one can prove the following relation,

$$\int_0^{2\pi} P_k(\Omega\bullet\Omega_k)d\varphi = 2\pi P_k(\Omega\bullet\hat{n})P_k(\hat{n}\bullet\Omega_k) \tag{24}$$

Plugging Eq. (24) into Eq. (23) and using the recursion relation we then get the following result,

$$\Phi_n = \sum_{k=even} A^n_k P_k(\hat{n}\bullet\Omega_k)\phi_k(r) \tag{25}$$

$$A^n_k = \frac{1}{2}\int_0^1 P_n(x)[kP_{k-1}(x) + (k+1)P_{k+1}(x)]dx \tag{26}$$

Note that in Eq. (26) although we have products of an even order Legendre polynomial with an odd order Legendre polynomial, the orthogonal relation for Legendre polynomials does not apply because the integration limit covers only half of its argument space. Also note that the even order Legendre polynomial in Eq. (25) approaches unity as its argument approaches unity, which happens when Ω_k is parallel to $\hat{n}$.

Combining Eq. (18), Eq. (22) and Eq. (25), we have the results for all the moments of the partial currents. It must be emphasized that the moments of the partial currents contain coefficients that depend on the inner product $\hat{n} \cdot \Omega_k$. This will result in non-linearity in the boundary conditions because the direction vector Ω_k is unknown until the SP_n equations are solved.

For the special case of SP_3, setting *n=0 or 2* in the above generic results gives,

$$J^{\pm}_0 = \frac{1}{4}\phi_0(r) + \frac{5}{16}P_2(\hat{n}\bullet\Omega_2)\phi_2(r) \mp \frac{1}{6\Sigma_t}[\hat{n}\bullet\nabla\phi_0(r) + 2\hat{n}\bullet\nabla\phi_2(r)] \tag{27}$$

$$\begin{aligned} J^{\pm}_2 = &[\frac{1}{16}\phi_0(r) + \frac{5}{16}P_2(\hat{n}\bullet\Omega_2)\phi_2(r)] \\ &\mp \frac{1}{10\Sigma_t}\{\frac{2}{3}\hat{n}\bullet\nabla\phi_0(r) + [\frac{4}{3} + \frac{9}{7}\frac{P_3(\hat{n}\bullet\Omega_2)}{\hat{n}\bullet\Omega_2}]\hat{n}\bullet\nabla\phi_2(r)\} \end{aligned} \tag{28}$$

If the non-linear effect is ignored by assuming $\hat{n}$ parallel to Ω_2, Eq. (27) and Eq. (28) then reduce to the familiar conventional SP_3 result.

2.4.2 The internal interface boundary condition

On an internal interface, the moments of partial currents are required to be continuous. This can be satisfied with the following conditions,

$$\phi_n(r)\big|_{Left} = \phi_n(r)\big|_{Right} \tag{29}$$

$$\frac{1}{\Sigma_t}\nabla\phi_n(r)\bigg|_{Left} = \frac{1}{\Sigma_t}\nabla\phi_n(r)\bigg|_{Right} \tag{30}$$

$$\Omega_n\big|_{Left} = \Omega_n\big|_{Right} \tag{31}$$

Eq. (31) is required to make the non-linear coefficients continuous on the interface. However the current continuity in Eq. (30) implies the continuity of the direction of the gradient vector, although the magnitude of the gradient may not be continuous. Hence Eq. (31) is automatically satisfied and the non-linear factors cancel out on the two sides of the interface. Therefore we conclude that the generic interface boundary conditions for SP_n, regardless of the order n, are the same as the conventional familiar ones, i.e. the continuity of flux moment and current moment.

2.4.3 The external boundary condition

The external boundary condition can be expressed in terms of the reflection albedo defined as the ratio of the incoming to the outgoing angular flux,

$$J_n^- = \alpha J_n^+ \tag{32}$$

Eq. (32) provides a set of equations relating the gradients of flux moments to the flux moments on the external boundary. The coefficients in the equations contain the non-linear factor $\hat{n}\cdot\Omega_k$ in the argument of Legendre polynomials. If the gradient vector is parallel to the normal direction, then all the Legendre polynomial factors reduce to unity and the external boundary condition reduces to the conventional one.

As a specific example, from Eq. (27) and Eq. (28) we get the vacuum boundary condition for SP_3 as follows.

$$-\frac{1}{3\Sigma_t}[\hat{n}\bullet\nabla\phi_0(r) + 2\hat{n}\bullet\nabla\phi_2(r)] = \frac{1}{2}\phi_0(r) + \frac{5}{8}P_2(\hat{n}\bullet\Omega_2)\phi_2(r) \tag{33}$$

$$-\frac{3}{7\Sigma_t}\frac{P_3(\hat{n}\bullet\Omega_2)}{\hat{n}\bullet\Omega_2}\hat{n}\bullet\nabla\phi_2(r) = -\frac{1}{8}\phi_0(r) + \frac{5}{8}P_2(\hat{n}\bullet\Omega_2)\phi_2(r) \tag{34}$$

It is noted that $P_2(\mu)$ decreases from 1 to -0.5 and $P_3(\mu)/\mu$ decreases from 1 to -1.5 as μ varies from 1 to 0.

3. Concluding remarks and a test problem

The proposed SP_n theory can provide the explicit angular flux reconstruction. The new theory results in all the same equations as the conventional SP_n equations except for the ones of the external boundary conditions. In case of $\hat{n} = \Omega_k$, the external boundary condition becomes the same as that in the conventional SP_n theory. Generally the flux gradient on the external boundary surface is not perpendicular to the surface. In practice one can assume the conventional external boundary condition to start the SP_n calculation. The external boundary condition can then be updated to iterate the calculation once more to estimate the correction effect. It would be interesting to see if the correction would be in the right direction.

The following simple problem is suggested to test the proposed SP_n theory. Consider a 2D square domain of homogeneous composition with vacuum boundary condition. Let the neutron scattering be mono-energetic and isotropic. The reference solution can be numerically generated with a transport code. The problem can be easily solved in diffusion theory, and in the conventional SP_3 theory as well. Now we consider how to solve it with the new SP_3 equations. The only difference is in the use Eq. (33) and Eq. (34). The conventional SP_3 theory will set $\mu = \hat{n}\cdot\Omega_2 = 1$. While for the new SP_3 theory, μ will vary along the boundary of the square domain. By the symmetry of the problem, we know that the angle between $\hat{n}$ and Ω_2 continuously varies from 0-degree at the center of the boundary surface to 45-degree at the corner of the square. Thus μ will vary from 1 to $\frac{1}{\sqrt{2}}$. Thus at the corner the value of the non-linear factors are,

$$P_2\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}\ ; \qquad \frac{P_3\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}}} = -\frac{1}{4}$$

It should be a reasonable initial guess that the variation of μ is approximately parabolic, symmetric about the center of the boundary surface. The solution of the new SP_3 theory should be quite different from that of the conventional one around the corner of the square. The angular flux reconstructed from the new SP_3 solution can be compared to the reference angular flux in the corner area. The vacuum boundary condition can be generalized to the generic albedo condition, where the above analysis of how μ varies on the boundary surface still applies. The above calculation could be extended from the new SP_3 theory to the new SP_n theory for higher orders to see if there could be any further improvement. Also the suggested problem can include both cases of a fixed source problem and an eigenvalue problem. Interested readers are highly encouraged to work out this numerical test problem.

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