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The Extinction Probability in Systems Randomly Varying in Time

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Abstract - *The extinction probability of a branching process (a neutron chain in a multiplying medium) is calculated for a system randomly varying in time. The evolution of the first two moments of such a process were calculated previously by the authors in a system randomly shifting between two states of different multiplication properties. The same model is used here for the investigation of the extinction probability. It is seen that the determination of the extinction probability is significantly more complicated than that of the moments, and it can only be determined by pure numerical methods. The dependence of the extinction probability on the properties of the randomly fluctuating system are in agreement with the findings regarding the moments of the population, but appear to show some new features as well. The results bear some significance not only for neutron chains in a multiplying medium, but also for evolution of biological populations in a time-varying environment.*

I. INTRODUCTION

This paper discusses some aspects of the calculation of the extinction probability in settings other than the classical case of the extinction of family trees or neutron chains in a stationary multiplying medium. The setting discussed here is the extinction probability in systems randomly varying in time. Such systems were studied before [1, 2, 3], but only the temporal evolution of the first two moments was investigated. As will be seen, the calculation of the extinction probability is a considerably more complicated task, which necessitates the use of numerical methods.

This work is strongly related to the collaboration of the author with two of his colleagues, namely Lénard Pál and Mike Williams. Hence at the end of the paper also some personal reminiscences and reflections on the history of these collaborations are mentioned. In particular some subtleties of the properties of branching processes in temporally randomly varying media, which are relevant to the present work, and which were discovered together with L. Pál, are described.

II. THEORY

Ever since the classic work of Galton and Watson on the extinction of family trees, the extinction probability of a branching process, started by one entity (individual/particle), has always been derived from a backward type master equation. One can write down a backward master equation for the generating function $g(z,t)$ of the probability distribution $p(n,t)$,

$$g(z,t) = \sum_{n=0}^{\infty} z^n \, p(n,t) \tag{1}$$

of having n particles in the system at time t, given that at $t = 0$ there was one neutron in the system as [3]

$$\frac{\partial g(z,t)}{\partial t} = Q \left\{ q\left[g(z,t)\right] - g(z,t) \right\} \tag{2}$$

with the initial condition

$$g(z,0) = z. \tag{3}$$

Here, Q is the intensity of the reaction, and $q(z)$ is the generating function of the probability distribution $f(n)$ of having n particles from a reaction, i.e.

$$q(z) = \sum_{n=0}^{\infty} z^n \, f(n) \tag{4}$$

From this, it is immediately possible to obtain an equation for the probability $p(0,t) \equiv p_0(t)$ of extinction until time t, since $p_0(t) = g(0,t)$. The extinction probability

$$p_0 = \lim_{t\to\infty} p_0(t)$$

is obtained immediately from (2) by assuming $dp_0(t)/dt = 0$ when $t \to \infty$, as the root of the equation

$$q(p_0) = p_0 \tag{5}$$

Actually, the above equation can be derived directly from a backward-type reasoning, considering the possible fate (=reaction) of the first individual (particle). The reasoning, due to the Dane Agner Krarup Erlang, a member of the famous Krarup family by his mother, which was about to become extinct, was published in the Matematisk Tiddskrift in 1929 and it goes as follows. The extinction probability p_0 is equal to the sum of the probabilities of the mutually exclusive events that the first particle either will not have any secondaries, with probability f(0), or will have one descendant, with probability $f(1)$, which will have to die out (with probability p_0), or will have two descendants (probability $f(2)$) which both will have to die out (probability p_0^2) etc. That is,

$$p_0 = f(0) + f(1)\, p_0 + f(2)\, p_0^2 + \cdots = q(p_0) \tag{6}$$

Hence one can derive an equation directly for the extinction probability, without the need of first deriving an equation for $g(z,t)$ and then substitute $z = 0$ and take the limit $t \to \infty$.

However, for systems varying randomly in time, the backward equation is not applicable. The main reason, as discussed in [1, 2, 3], is that the factorisation ansatz of the backward equation cannot be applied, because the evolution of the chains started by neutrons born simultaneously will not be independent (will be influenced simultaneously by the changing properties of the material). Hence the only possibility for the calculation of the extinction probability in a temporally randomly varying medium is to use the forward equation.

Application of the forward equation for the determination of the extinction probability is though much more cumbersome than that of the backward equation, and is not to be found in the literature. The difficulties will be illustrated with the case of the traditional branching process in a stationary medium. The forward equation for this case reads as

$$\frac{\partial g(z,t)}{\partial t} = Q[q(z) - z]\,\frac{\partial g(z,t)}{\partial z}. \tag{7}$$

Substituting $z = 0$ in Eq. (7) shows that there is a closure problem; the resulting equation contains both $p_0(t)$ and $p_1(t)$; differentiation with respect to z and substituting $z = 0$ will lead to and equation containing $p_1(t)$ and $p_2(t)$, and so on. Besides, being a forward equation, operating on the final co-ordinates, one cannot take the limit $t \to \infty$ in the equation itself, only in the solution.

It is also obvious that heuristic reasoning of the type of Eq. (6) is not possible either; the asymptotic value of p_0 requires the knowledge of the asymptotic value of p_1, which requires the knowledge of p_2 and so on, illustrating again the problem of closure and the knowledge of the full solution of $p(n,t)$.

The above shows that a suitable starting point is to first investigate the possibilities of determining the extinction probability from a forward master equation for the classic case of the static medium, after which the solution method may be attempted to be generalised to the case of the randomly varying system. Two basic possibilities appeared to be worth trying. The first is to Laplace transform in time the forward equation, and then seek the asymptotic value of the extinction probability with the help of the Tauberian theorem. In other words, taking the Laplace transform

$$\bar{g}(z,s) = \int_0^\infty e^{-st} g(z,t)dt \tag{8}$$

will convert Eq. (7) to

$$s\,\bar{g}(z,s) - z = Q[q(z) - z]\,\frac{\partial \bar{g}(z,s)}{\partial z}, \tag{9}$$

This differential equation may be solved for $g(z,s)$. Since the extinction probability is defined as

$$p_0 = \lim_{t\to\infty} g(z = 0, t), \tag{10}$$

From the solution for $g(z,s)$, this can be recovered as

$$p_0 = \lim_{s\to 0} s\,\bar{g}(z = 0, s) \tag{11}$$

If such a solution can be obtained, one might try to generalise it to the case of a medium randomly varying in time.

If this method should not work, then one can restrict the branching process to a quadratic process, such that the total number of new-born particles is either zero, one, or at most two, i.e.

$$f(n) = f_0\,\delta_{n,0} + f_1\,\delta_{n,1} + f_2\,\delta_{n,2} \tag{12}$$

and hence

$$q(z) = f_0 + f_1\,z + f_2\,z^2 \tag{13}$$

It was shown in [3] that in a static system, for such a case a complete time-dependent solution can be obtained for the full generating function, and hence for the extinction probability. Hence this method appeared to have larger potential to be applicable for the randomly varying system.

In the case of a system randomly varying between two states, such as in those treated in [3], the notations of which will be used here, one seeks the generating function $g_{j,i}(z,t)$, $\{i, j = 1, 2\}$ of the probability that at time t there are n particles in the system, and the system is in state j, on the condition that at time $t = 0$, the system was in state i and there was one particle in the multiplying medium. Since we are only interested in the asymptotic behaviour of the neutron population irrespective of the state of the system, we seek the extinction probability

$$p_{0,i} = \lim_{t\to\infty}\,[g_{1,i}(0,t) + g_{2,i}(0,t)] \tag{14}$$

It is this quantity whose calculation is attempted in this paper.

As will be seen, none of the above expectations worked, and only a pure numerical scheme made it possible to calculate the extinction probability in systems randomly varying in time.

III. SOLUTIONS

1. General solution in a static medium

We start with the forward equation

$$\frac{\partial g(z,t)}{\partial t} = Q\,[q(z) - z]\,\frac{\partial g(z,t)}{\partial z}, \tag{15}$$

which can be re-written by introducing $\tau = Qt$ as

$$\frac{\partial g(z,\tau)}{\partial \tau} = [q(z) - z]\,\frac{\partial g(z,\tau)}{\partial z} \tag{16}$$

Taking a Laplace transform in time we have

$$\bar{g}(z,s) = \int_0^\infty d\tau e^{-s\tau} g(z,\tau) \tag{17}$$

from which can deduce the following condition which may be of use later, viz:

$$\underset{\lim\ s\to 0}{s\bar{g}(z,s)} = g(z,\infty) \tag{18}$$

Then Eq. (16) becomes

$$(q(z) - z)\,\frac{d\bar{g}}{dz} = s\bar{g} - z \tag{19}$$

Rearranging this we have

$$\frac{d\bar{g}}{dz} - \frac{s}{q(z) - z}\bar{g} = -\frac{z}{q(z) - z} \tag{20}$$

Introducing the integrating factor leads to

$$\frac{d}{dz}\left(\bar{g}e^{-s\int^{z}\frac{dz'}{q(z')-z'}}\right) = -\frac{z}{q(z) - z}e^{-s\int^{z}\frac{dz'}{q(z')-z'}} \tag{21}$$

Integrating from z to unity, we find

$$\bar{g}(1, s)e^{-s\int^{1}\frac{dz'}{q(z')-z'}} - \bar{g}(z, s)e^{-s\int^{z}\frac{dz'}{q(z')-z'}} = -\int_{z}^{1}\frac{z'dz'}{q(z') - z'}e^{-s\int^{z'}\frac{dz''}{q(z''')-z''}} \tag{22}$$

After some re-arrangement this becomes

$$\bar{g}(z, s) = \frac{1}{s}e^{-s\int_{z}^{1}\frac{dz'}{q(z')-z'}} + \int_{z}^{1}\frac{z'dz'}{q(z') - z'}e^{-s\int_{z}^{z'}\frac{dz''}{q(z''')-z''}} \tag{23}$$

where we have set $\bar{g}(1, s) = 1/s$. Again we may write

$$s\bar{g}(z, s) = e^{-s\int_{z}^{1}\frac{dz'}{q(z')-z'}} + s\int_{z}^{1}\frac{z'dz'}{q(z') - z'}e^{-s\int_{z}^{z'}\frac{dz''}{q(z''')-z''}} \tag{24}$$

We may now be tempted to use the relation Eq. (18). However this leads on first sight to $g(z, \infty) = 1$, for all z, which is not helpful. The fact that this result occurs means that we have not dealt with singularities appearing in the integrands of eq (9). These occur at the zeros of

$$q(z) - z = 0 \tag{25}$$

It would appear that before taking the limit in Eq. (24) we should use the properties of $q(z)$. Indeed, if one specifies the process as quadratic, i.e. only absorption, scattering and binary fission can take place, the problem can be solved. This was shown in the book. Now we restrict ourselves to such processes, and turn to the binary random medium.

2. Qadratic process in a time-varying medium

We will now employ the quadratic process, characterised with the number distribution (12) and its quadratic generating function (13). Note that since

$$\sum_{n} f(n) = 1, \tag{26}$$

one of the f_i can be expressed with the other two, which simplifies the notations. As is known, in the classical case, the extinction probability is unity for subcritical and critical systems, i.e. when

$$\mathbf{E}\{n\} = \left[\frac{dq(z)}{dz}\right]_{z=1} = q'(1) \equiv \nu \leq 1, \tag{27}$$

whereas it becomes less then unity for $\nu > 1$. From (12) or (13) one obtains

$$\nu = f_1 + 2f_2 \tag{28}$$

using (26) leads to the condition of criticality as

$$f_0 = f_2. \tag{29}$$

The distribution can be defined by two parameters, e.g. by f_0 and f_2, $f_0 + f_2 \leq 1$, or by the mean ν and either the variance or the second factorial moment $q_2 \equiv q''(1)$ of the number of secondary particles per reaction.

The temporally randomly varying multiplying system will be the same as that used in our previous work [1, 2, 3]. It is assumed that the system has two states, with reaction intensities Q_i and the generating functions of the distribution neutrons per reaction, $q_i(z)$, $i = 1, 2$. The generating functions will be defined in terms of the parameters ν_i and $q_{2,i}$, $i = 1, 2$. We will seek the generating functions $g_{j.i}(z.t)$, $j, i = 1, 2$ of the probability that at time t, the system is in state j and the number of neutrons in the system equals n, given that at time $t = 0$ the system was in state i and there was one neutron in the system. It is also assumed that the probability that during time Δt the system changes from state 1 to state 2, or vice versa, is equal to $\lambda\Delta t + o(\Delta t)$.

As it is shown e.g. in [3], the generating functions $g_{j.i}(z.t)$ obey the following coupled differential equation system:

$$\frac{\partial g_{1,i}(z, t)}{\partial t} = Q_1 \left[q_1(z) - z\right] \frac{\partial g_{1,i}(z, t)}{\partial z} + \lambda \left[g_{2,i}(z, t) - g_{1,i}(z, t)\right], \tag{30}$$

and

$$\frac{\partial g_{2,i}(z, t)}{\partial t} = Q_2 \left[q_2(z) - z\right] \frac{\partial g_{2,i}(z, t)}{\partial z} + \lambda \left[g_{1,i}(z, t) - g_{2,i}(z, t)\right], \tag{31}$$

with the initial conditions

$$g_{j,i}(0, z) = z\,\delta_{ij}, \qquad i, j = 1, 2.$$

The expectation is that substituting quadratic forms for the $q_i(z)$, taking temporal Laplace transforms and eliminating, say, $g_{2,i}(z, s)$, a differential equation in z can be derived for $g_{1,i}(z, s)$. Having solved this differential equation, one can obtain the corresponding extinction probability by the Tauberian theorem as

$$\lim_{t\to\infty} g_{1,i}(0, t) = \lim_{s\to 0} s\, g_{1,i}(0, s) \tag{32}$$

Unfortunately, this strategy does not work because the arising differential equation for $g_{1,i}(z, s)$ is not amenable for an analytic solution. A Laplace transform in time yields

$$Q_1 \left[q_1(z) - z\right] \frac{\partial \bar{g}_{1,i}(z, s)}{\partial z} + \lambda\, \bar{g}_{2,i}(z, s) - \left[\lambda + s\right] \bar{g}_{1,i}(z, s) = z\,\delta_{1,i} \tag{33}$$

and

$$Q_2 \left[q_2(z) - z\right] \frac{\partial \bar{g}_{2,i}(z, s)}{\partial z} + \lambda \bar{g}_{1,i}(z, s) - \left[\lambda + s\right] \bar{g}_{2,i}(z, s) = z\,\delta_{2,i} \tag{34}$$

Substitution of a quadratic form for the $q_i(z)$ yields for the factors multiplying the derviatives in (33) and (34)

$$q_i(z) - z = (1-z)\left(1 - \bar{\nu}_i + \frac{1}{2}\, q_{2,i}(1-z)\right) \tag{35}$$

Putting these into the equations, differentiating (33) w.r.t. z and eliminating $g_{2,i}(t,z)$ leads to

$$\frac{d^2\bar{g}_1(z,s)}{dz^2} + \left[\frac{\nu_1 - 1}{q_1(z)-z} - \frac{\lambda+s}{q_1(z)-z} - \frac{\lambda+s}{q_2(z)-z}\right]\frac{d\bar{g}_1(z,s)}{dz} + \frac{s(s+2\lambda)}{(q_1(z)-z)\,(q_2(z)-z)}\bar{g}_1(z,s) + \frac{q_2(z) - z(2\lambda+1+s)}{(q_1(z)-z)\,(q_2(z)-z)} = 0 \tag{36}$$

This equations shows the basic difference between the determination of the moments and that of the extinction probability. When calculating the moments, the substitution $z = 1$ can be made already in the defining equations. Hence after a temporal Laplace transform, there remain only algebraic equations to be solved with constant coefficients for the $g^{(n)}_{j,i}(1,s)$. These can be readily handled, even with a general (not quadratic) $q_i(z)$, since only the moments of this distribution occur. For the extinction probability, the substitution $z = 0$ would lead to a closure problem, hence the differential equations first need to be solved for the $g_{j,i}(z,s)$, and the substitution $z = 0$ can only be made afterwards. The differential equation is not of constant coefficients, rather the coefficients are highly non-linear functions of z. This shows that the derivation of the extinction probability in a randomly varying medium is substantially more complicated than calculating the first two moments of the neutron distribution.

3. Numerical solution

Since there is very little hope that Eq. (35) can be solved analytically, we chose a numerical solution. Since a numerical solution is not suitable for the application of the Tauberian theorem, instead of solving the Laplace-transformed equations (33) - (34) or (36), the original equations (30) - (31) will be solved.

It is convenient to transform the variable $z\,(0,1)$ to $x\,(-1,1)$ to $x = 2z - 1$, so that it conforms to the space of the Chebyshev polynomials. Thus, Eqs (30) and (31) then become

$$\frac{\partial g_1(x,t)}{\partial t} = Q_1\,(1-x)\left(1 - \nu_1 + \frac{1}{4}q_{2,1}(1-x)\right)\frac{\partial g_1(x,t)}{\partial x} + \lambda\left[g_2(x,t) - g_1(x,t)\right] \tag{37}$$

and

$$\frac{\partial g_2(x,t)}{\partial t} = Q_2\,(1-x)\left(1 - \nu_2 + \frac{1}{4}q_{2,2}(1-x)\right)\frac{\partial g_2(x,t)}{\partial x} + \lambda\left[g_1(x,t) - g_2(x,t)\right] \tag{38}$$

and the initial conditions become

$$g_{1,i}(x,0) = \frac{1}{2}(1+x)\delta_{1i} \;\text{ and }\; g_{2,i}(x,0) = \frac{1}{2}(1+x)\delta_{2i}. \tag{39}$$

We solve equations (37) and (38) by replacing the derivative $\partial/\partial x$ by a Chebyshev-Gauss-Lobatto collocation in the form

$$\left.\frac{\partial y(x)}{\partial x}\right|_{x_k} \approx \sum_{j=0}^{N} D_{k,j}\, y(x_j) \tag{40}$$

where $x_j = \cos(\pi j/N)$ and the $D_{k,j}$ are defined in [4] (but see the original paper by Don and Solomonoff [5] for correct form).

$$\frac{dg_{1k,i}(t)}{dt} = Q_1(1-x_k)\left(1 - \nu_1 + \frac{1}{4}q_{2,1}(1-x_k)\right)\times \left[\sum_{j=1}^{N} D_{k,j}g_{1,j,i}(t) + D_{k,0}g_{1,0,i}(t)\right] + \lambda_1\left(g_{2,k,i}(t) - g_{1,k,i}(t)\right) \tag{41}$$

and

$$\frac{dg_{2k,i}(t)}{dt} = Q_2(1-x_k)\left(1 - \nu_2 + \frac{1}{4}q_{2,2}(1-x_k)\right)\times \left[\sum_{j=1}^{N} D_{k,j}g_{2,j,i}(t) + D_{k,0}g_{2,0,i}(t)\right] + \lambda_1\left(g_{1,k,i}(t) - g_{2,k,i}(t)\right) \tag{42}$$

Eqs (41) and (42), together with the initial and boundary conditions, allow a numerical solution, from which the extinction probability is extracted at $z = 0$ or $x = -1$. Since we are interested in the extinction probability irrespective of the system state, we use Eq. (14). In terms of the collocation points this is expressed as

$$g_i(0,t) = g_{1,N,i}(t) + g_{2,N,i}(t) \tag{43}$$

This way the whole time dependence of the probability $g_i(0,t)$ of no particles being in the system is recovered, and the extinction probability is equal to its value for a t for which the asymptotic state is reached.

4. Quantitative results and discussion

Quantitative results for some characteristic cases will be shown below. The selection of the cases is based on the knowledge gained through the study of the behaviour of the moments of the neutron population in systems randomly varying in time [1, 2, 3]. In these works, a system, both of whose states are subcritical or supercritical, are called strongly subcritical and strongly supercritical, respectively. Those systems which fluctuate between a subcritical and a supercritical state fall into three categories. Defining the parameters

$$\alpha_i = Q_i\,(\nu_i - 1), \tag{44}$$

and

$$\lambda_{cr} = \frac{\alpha_1\,\alpha_2}{\alpha_1 + \alpha_2} \tag{45}$$

it was found that systems that fulfil the condition

$$\lambda = \lambda_{cr} \tag{46}$$

such that $\alpha_1 \alpha_2 < 0$, are critical in the mean, in the sense that the expectation of the population is constant. Systems where $\lambda < \lambda_{cr}$ are supercritical in the mean, thas is the expectation of the population diverges in time, whereas if $\lambda > \lambda_{cr}$, the system is subcritical in the mean, that is the expectation dies out asymptotically. In particular, it was shown that for systems for which one has

$$\alpha_1 + \alpha_2 = 0 \quad \text{or} \quad \frac{\nu_1 + \nu_2}{2} = 1, \tag{47}$$

whose “average criticality” is zero in time in the traditional sense, are supercritical in the mean, except the pathological case when λ diverges.

Some quantitative results are shown below, and they will be discussed in light of the above. Fig.1 shows the quantities $g_i(0,t)$ for the case of a strongly subcritical system for $i = 1, 2$. In the numerical work, the data $Q_1 = 10^4 s^{-1}$, $Q_2 = 2 \times 10^4 s^{-1}$, $\lambda = 10^3 s^{-1}, \nu_1 = 0.7$, $\nu_2 = 0.9$, $q_{2,1} = q_{2,2} = 9.2$ were used. As is expected, both extinction probabilities tend to unity, but $g_1(0,t)$, which corresponds to the case when the system started from the deeper subcritical case, reaches the asymptotic value faster. This is in accordance with the findings on the expectations in [3], where also a dependence of the asymptotic values on the initial conditions was found, and which is clear intuitively.

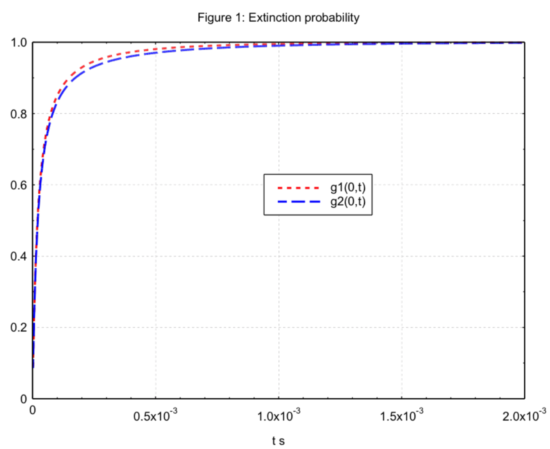

Fig. 1. Time dependence of the extinction probability in a strongly subcritical randomly varying system

In Fig.2 extinction probabilities are shown for the case of a strongly supercritical system with $\nu_1 = 2.4$ and $\nu_2 = 2$. The other data are the same as in the previous case. Here, as is also expected, both extinction probabilities are less then unity, with the system starting from the higher supercriticality (system 1) having a lower extinction probability. The asymptotic values of $g_1(0,t)$ and $g_2(0,t)$ are 0.7017 and 0.7761, respectively.

Next we consider a case when the system is fluctuating between a subcritical and a supercritical state with $\nu_1 = 0.6$ and $\nu_2 = 1.4$, with the values $\lambda = 10^3 s^{-1}$ and $Q_1 = Q_2 = 10^4 s^{-1}$. This system fulfils the condition expressed in (47), i.e. its average criticality is zero in the traditional sense. However, such systems were found to be supercritical in the mean. Indeed, as the results show in Fig. 3, the extinction probabilities are less then unity, the asymptotic values being 0.993 for system

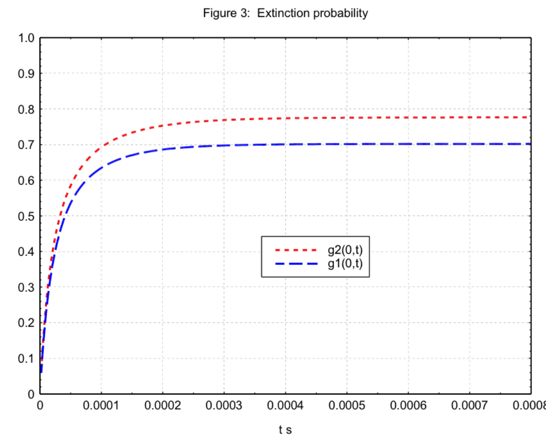

Fig. 2. Time dependence of the extinction probability in a strongly supercritical randomly varying system

1 starting from the subcritical case and 0.956 for system 2 starting from the supercritical case. The extinction probabilities are, however, much closer to unity than in the case of the strongly supercritical system. This is again in agreement with what one could expect from the behaviour of the moments in such systems.

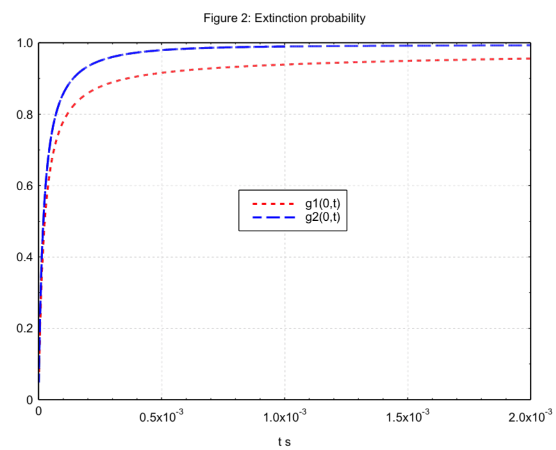

Fig. 3. Time dependence of the extinction probability in a randomly varying system which is supercritical in the mean

One further case of interest with a system supercritical in the mean is shown below. Fig. 4 shows the case with $\nu_1 = 1.2$ and $\nu_2 = 0.75$, $\lambda = 10^3\ s^{-1}$ and $Q_1 = Q_2 = Q = 10^4\ s^{-1}$. With these data one finds that $\lambda_{cr} = 10^4 s^{-1} > \lambda$, hence the system is still supercritical in the mean, although “less supercritical” than in the previous case, since if $\alpha_1 + \alpha_2 = 0$, λ_{cr} diverges. The asymptotic values are equal to $g_1(0,\infty) = 0.9774$ and $g_2(0,\infty) = 0.9997$, respectively. This latter figure differs from unity only in the range of the numerical uncertainties. Hence these numerical results indicate the interesting finding that for this particular system, which is supercritical in the mean, the extinction probability is less than unity only in the case when the system starts from the supercritical state. Starting from the subcritical state, the extinction probability appears to be very close, or equal to unity. This indicates that in this case the character of the extinction probability is principally different from that of the moments, for which the difference in the initial conditions did not lead to such a substantial difference. It has though to be added that these results are based on numerical results, and further checks are necessary to confirm the numerical accuracy of the present calculations.

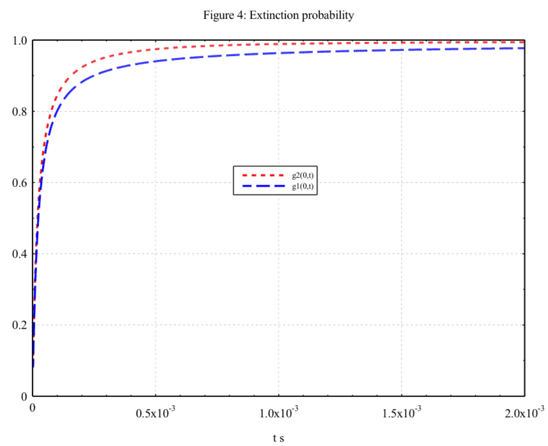

Fig. 4. Time dependence of the extinction probability in a randomly varying system which is "slightly supercritical" in the mean

It is now interesting to investigate the case when the system is exactly critical in the mean, according to the definitions given in [3], i.e. when Eqs (45) and (46) are fulfilled. This definition is based on the fact that in such a system the expectation of the population number is asymptotically constant, irrespective of the initial conditions (although the quantitative value of the asymptotic expectations does depend on the initial state). Based on the properties of the extinction probability in constant systems, where the extinction probability is unity in a critical system, it is expected that the extinction probabilities will tend to unity also in the time-varying system, irrespective of the initial conditions.

Such a case is shown in Fig. 5. With $Q_1 = Q_2 = Q = 10^4\ s^{-1}$ and $\lambda = 10^3\ s^{-1}$, the values $\nu_1 = 1.02$ and $\nu_2 = 0.975$ fulfil (45) and (46). Indeed, as the Figure shows, and also the numerical values confirm to four significant digits, the extinction probabilities tend to unity even in this case. However, the convergence is much slower than in the strongly supercritical system or in a system which is supercritical in the mean (note the logarithmic scale on the x-axis).

It is thus seen that the properties of the extinction probability in systems fluctuating in time are in agreement with those of the lower order moments, but appear to show some novel features as well. It is also seen that these properties can only be explored by numerical methods. The investigations are going on and will be extended to further cases and with extended convergence tests to both confirm the present results, as well as to get a deeper insight into the characteristics of the extinction probability in systems randomly varying in time.

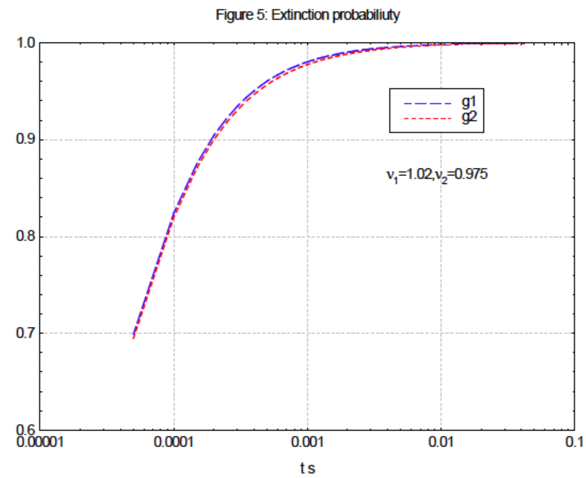

Fig. 5. Time dependence of the extinction probability in a randomly varying system which is critical in the mean

IV. SOME HISTORY

It is a special pleasure and privilege that this paper is the joint work with two senior colleagues and former mentors, true giants of reactor physics and neutron fluctuations, Lénárd Pál and Mike Williams. Besides personal friendship, scientific collaboration with them was thoroughly decisive in shaping my whole career. This paper is also historic in the sense that it is the first (but hopefully not the last) joint publication of the three of us. However, let me start with some preliminaries to this particular paper.

From the early days of my first encounter with neutron fluctuations in multiplying systems, I was fascinated by the complete separation of zero power noise and power reactor noise. Not only that they are valid in opposite limits of the neutron population (low and high neutron population, respectively), also their mathematical description (master equations for the probability distributions and the Langevin equation for the neutron density as a random process, respectively), as well as the use of the theory (determining the multiplication properties of the medium and parameters of the fluctuations of the medium, respectively), are all different. I found the usual explanations for the complete lack of contact between these two areas in the literature insufficient and not satisfactory, because they were alluding as if any such contact was impossible.

I became interested in trying to bridge these two areas by seeking a description which encompasses both types of neutron fluctuations, having zero power noise and power reactor noise as opposite limits of the theory. In particular I also envisaged that if any new book on neutron fluctuations was to be written, which I myself had some vague plans for, a discussion and clarification of this question must be included. The first occasion for taking up this line was in 1999, when I had a short sabbatical, during which I visited Anil Prinja at the UNM for less than a month. For simplicity, as a starting point, we assumed a multiplying medium with two different states, such that it was jumping randomly between those two states, and applied the master equations of zero power noise. A solution of this problem would contain both the effects of branching (zero power noise) and the fluctuations of the medium (power reactor noise). With the help of the backward master equation we managed to derive the first two moments of the neutron population, including the temporal covariance, and were able to recover the zero power and power reactor cases in the corresponding limits.

The story could have ended here. However, I was interested in extending the treatment from the case of a dichotomic random material (jumping randomly between two states) to a "real" random material, like the one that one always treats in traditional power reactor noise, i.e. a medium whose parameters change in time as a continuous random function, as opposed to a binary random function. Since I started collaborating with Lénárd Pál after 2002 (when he was a keynote speaker at the last SMORN symposium ever, held in Göteborg in May 2002), I approached him with the question of how to extend the case of a binary medium to a continuously varying medium. Lénárd first wanted to reconstruct our previous results by own calculations, and in his careful and general manner, just out of interest, solved the case of the binary random medium with both the forward and the backward method. To our surpr Springerise, the results, which we expected to be equivalent, differed from each other. After a long period of searching for an error in the calculations, we realised that the reason of the differing results is the fact that the backward equation is not applicable, due to the failure of the factorisation assumption which the backward equation requires in order to avoid the closure problem.

Thereafter we switched to the forward approach, and derived analytic solutions for the first two moments and the temporal covariance function of the neutron population both for one starting particle as well as for the case of a stationary external source. These results became a separate chapter in our book on neutron fluctuations [3]. However, we have not considered the extinction probability. As the above considerations show, this does not arise automatically from the solutions we obtained earlier, rather it constitutes an interesting challenge, which is tackled in this contribution.

Mike comes in to this work on another line, which is directly related to the extinction probability. First of all, my very first paper on master equations of particle transport was a joint publication we made during my IAEA postdoc fellowship at the Department of Nuclear Engineering of Queen Mary College, London, where Mike was my host. We published a paper on the statistics of radiation damage, actually a process in which a quadratic generation function is exact since there are at most two particles in one collision, the projectile and the recoil. This was way back in 1979, the paper appeared in J. Phys. D. in 1980. Strangely, however, although we have kept a close contact ever since then, we did not publish anything jointly until 2014, when Mike involved me into a study he made on the behaviour of the extinction probability in a system when the delayed neutrons are also accounted for. His interest in this subject was raised by some parts of our book with Lénárd on neutron fluctuations. Mike's experience with numerical treatment of the extinction probability became essential in tackling the problem treated in this case. I am thoroughly happy and grateful for having the privilege of doing some joint work with Lénárd and Mike, which converged into this joint paper.

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