Theoretical Discussion of Statistical Error for Variance-to-Mean Ratio

Tomohiro Endo[†], Akio Yamamoto

Graduate School of Engineering, Nagoya University: Nagoya, Japan, {t-endo, a-yamamoto}@nucl.nagoya-u.ac.jp

Abstract – *In the present paper, statistical error of variance-to mean ratio, or Y value in the Feynman-α method, is theoretically investigated to discuss the relationship among the statistical error of Y value, external neutron source strength, and measurement time. Practical theoretical formulae are derived to estimate the statistical error of Y value from a single measurement of reactor noise. The derived formulae clarify that the statistical error of Y value can be reduced by the total number of counting gate, or total measurement time, rather than the strength of external neutron source. Through an actual reactor noise experiment at the Kyoto University Criticality Assembly, the derived estimation formulae are validated.*

I. INTRODUCTION

Subcriticality monitoring is one of the important researches in order to achieve safe and efficient operation and management in nuclear fuel-related facilities. It is also important for the Accelerator-Driven System (ADS), where the subcritical state must be kept in operation [1]. Furthermore, in the retrieval of fuel debris from Fukushima Daiichi units 1-3 with the submersion condition, there is a possibility of positive reactivity insertion event due to the change of the moderation ratio; thus the subcriticality monitoring to prevent the recriticality is one of the important issues [2,3].

The Feynman-α method, or the variance-to-mean ratio method, is one of the practical subcriticality measurement techniques on the basis of "zero-power reactor noise analysis" [4-7]. Using the Feynman-α method, the prompt neutron decay constant α can be measured by analyzing time-series data of neutron counts, then the measurement value of α is converted to the subcriticality $-\rho$, which is the absolute value of negative reactivity. In the Feynman-α method, quantification of statistical error of the variance-to-mean ratio, or Y value, is useful information to determine the measurement time depending on the neutron count rate level. Here, one of the simple estimation methods for the statistical error is multiple measurements of reactor noise; however, it requires longer measurement time to repeat the multiple times of measurements. Thus, in authors' previous study, statistical error estimation technique using only a single measurement of reactor noise, *i.e.* without multiple measurements, was proposed by the aid of the bootstrap method, which is one of the resampling techniques [8,9].

As another approach, the present paper newly proposes theoretical formulae to estimate the statistical error of Y value for a single measurement. The motivation is to clarify a major factor in the statistical error of Y value. In the following Sec. II, the theory for statistical error of Y value is described. Through an analysis for actual reactor noise data which were measured at the Kyoto University Criticality Assembly (KUCA), the derived estimation formulae are demonstrated in Sec. III, where the estimated statistical errors using the derived formulae are compared with (1) the bootstrap statistical error [9] and (2) the reference value from multiple measurements. Finally, Sec. IV presents the concluding remarks.

II. THEORY

1. Fundamental Theory for Statistical Error of Y value

Let us assume a steady state of source-driven subcritical system. In this subcritical system, neutron counts $C_i(T)$ ($i = 1 \sim N$) are measured multiple times, where T is a counting gate width and N is the total number of count data. Then, Y value is evaluated as the variance-to-mean ratio:

$$Y \equiv \frac{\sigma^2}{\langle C \rangle} - 1 \approx \frac{s^2}{\bar{C}} - 1, \tag{1}$$

$$\sigma^2 \equiv \langle (C - \langle C \rangle)^2 \rangle, \tag{2}$$

where the bracket $\langle\,\rangle$ means the expected value; $\langle C \rangle$ and σ^2 are the population mean and variance of neutron counts; and $\bar{C}$ and s^2 represent the sample mean and the unbiased variance of $C_i(T)$, respectively:

$$\bar{C} = \frac{1}{N}\sum_{i=1}^{N}(C_i - \bar{C})^2, \tag{3}$$

$$s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(C_i - \bar{C})^2. \tag{4}$$

Note that the notation T is omitted in Eqs. (1)-(4) for simplicity. Based on the propagation of uncertainty (or the sandwich rule) for Eq. (1), the statistical error of Y value (hereafter denoted as σ_Y) can be estimated as follows:

$$\sigma_Y \approx (1+Y)\sqrt{\left(-\frac{\sigma_{\bar{C}}}{\bar{C}}\right)^2 + \left(\frac{\sigma_{s^2}}{s^2}\right)^2 - 2\frac{\mathrm{cov}(\bar{C}, s^2)}{\bar{C}s^2}}, \tag{5}$$

where $\sigma_{\bar{C}}$ and σ_{s^2} mean the statistical errors of $\bar{C}$ and s^2, respectively; and $\mathrm{cov}(\bar{C}, s^2)$ is the covariance between $\bar{C}$ and s^2. In Eq. (5), the expected values of $\sigma_{\bar{C}}$, σ_{s^2}, and $\mathrm{cov}(\bar{C}, s^2)$ can be derived as follows [10]:

$$\langle\sigma_{\bar{C}}\rangle = \sqrt{\frac{\sigma^2}{N}}, \tag{6}$$

$$\langle\sigma_{s^2}\rangle = \sqrt{\frac{1}{N}\left(\mu_4 - \frac{N-3}{N-1}(\sigma^2)^2\right)}, \tag{7}$$

$$\langle\mathrm{cov}(\bar{C}, s^2)\rangle = \frac{\mu_3}{N}, \tag{8}$$

$$\mu_3 \equiv \langle(C - \langle C\rangle)^3\rangle, \tag{9}$$

$$\mu_4 \equiv \langle(C - \langle C\rangle)^4\rangle, \tag{10}$$

where μ_3 and μ_4 correspond to the 3rd and 4th order central moments, respectively.

2. Theoretical Expression for Poisson distribution

In order to gain more insight about the statistical error of Y value, let us consider that the probabilistic distribution of neutron counts can be well approximated by the Poisson distribution. For example, this condition corresponds to the situation where there are no fissile materials and the external neutron source emits only one neutron per decay (Poisson source). As another example, if the subcriticality is deep and/or the detection efficiency is very low, the histogram of neutron counts may be well approximated by the Poisson distribution. Based on the characteristics of the Poisson distribution, σ^2, μ_3 and μ_4 satisfy the following relationships:

$$\sigma^2 = \lambda, \tag{11}$$

$$\mu_3 = \lambda, \tag{12}$$

$$\mu_4 = \lambda + 3\lambda^2, \tag{13}$$

where λ corresponds to the mean of neutron count, *i.e.* $\lambda = \langle C\rangle$. By substituting Eqs. (11)-(13) into Eqs. (6)-(8), the expected value of σ_Y can be obtained from Eq. (5) as follows:

$$\begin{aligned}\sigma_{Y,\mathrm{P}} &\approx \sqrt{\left(-\sqrt{\frac{1}{N\lambda}}\right)^2 + \left(\sqrt{\frac{1}{N\lambda} + \frac{2}{N-1}}\right)^2 - \frac{2}{N\lambda}} \\ &= \sqrt{\frac{2}{N-1}},\end{aligned} \tag{14}$$

where $\sigma_{Y,\mathrm{P}}$ means the expected value of statistical error of Y value in the case of Poisson distribution.

If a high detection efficiency detector and/or a high strength external neutron source is used for the Feynman-α experiment, the relative statistical error of sample mean, *i.e.* $\langle\frac{\sigma_{\bar{C}}}{\bar{C}}\rangle \approx \frac{1}{\sqrt{N\lambda}}$, can be reduced inversely proportional to the square root of λ. It also means that the total measurement time, or NT, can be reduced in such a situation, if the same statistical error level for $\langle\frac{\sigma_{\bar{C}}}{\bar{C}}\rangle$ are desired.

On the hand, Eq. (14) clarifies that $\sigma_{Y,\mathrm{P}}$ can be reduced by only increasing the total number of count data N, due to the relative statistical error of $\langle\frac{\sigma_{s^2}}{s^2}\rangle \approx \sqrt{\frac{1}{N\lambda} + \frac{2}{N-1}}$. It is interestingly noted that the relative covariance term $\langle\frac{\mathrm{cov}(\bar{C}, s^2)}{\bar{C}s^2}\rangle \approx \frac{1}{N\lambda}$ is not negligible if λ is close to zero (*e.g.* the neutron count rate is low and/or the counting gate width T is small), because the correlation coefficient $\langle\frac{\mathrm{cov}(\bar{C}, s^2)}{\sigma_{\bar{C}}\sigma_{s^2}}\rangle \approx 1/\sqrt{1 + \frac{2N}{N-1}\lambda}$ converges to +1 (*i.e.* strongly positive correlation coefficient) as λ approaches zero. Thus, this fact suggests that the correlation between $\bar{C}$ and s^2 should be taken into account in the estimation of the statistical error of Y value.

3. Practical Estimation Method

A. Estimation Formula using Unbiased Estimators for Central Moments

In an actual reactor noise experiment, the probability distribution of neutron count does not necessarily satisfy the Poisson distribution because of the neutron-correlation due to multiple emission of neutrons in the decay of external source and in the fission event. This physical phenomenon is the measurement principle of reactor noise analysis such as the Feynman-α method. Then, σ^2, μ_3 and μ_4 can be estimated using the unbiased estimators s^2, M_3 and M_4, respectively [11]; where M_3 and M_4 are obtained by

$$M_3 = \frac{N}{(N-1)(N-2)}\sum_{i=1}^{N}(C_i - \bar{C})^3, \tag{15}$$

$$\begin{aligned}M_4 &= \frac{N^2 - 2N + 3}{(N-1)(N-2)(N-3)}\sum_{i=1}^{N}(C_i - \bar{C})^4 \\ &\quad - \frac{3(2N-3)}{N(N-1)(N-2)(N-3)}\left(\sum_{i=1}^{N}(C_i - \bar{C})^2\right)^2.\end{aligned} \tag{16}$$

Using these unbiased estimators s^2, M_3 and M_4 in Eqs. (5)-(8) instead of σ^2, μ_3 and μ_4, the statistical error $\sigma_{Y,\mathrm{est}}$ can be evaluated by the following formula:

$$\begin{aligned}&\sigma_{Y,\mathrm{est}} \\ &\approx (1+Y)\sqrt{\frac{1+Y}{N\bar{C}} + \frac{1}{N}\left(\frac{M_4}{(s^2)^2} - \frac{N-3}{N-1}\right) - \frac{2M_3}{N\bar{C}s^2}}.\end{aligned} \tag{17}$$

In the conventional Feynman-α method, the calculation of $\bar{C}$ and s^2 are required for the evaluation of Y value as defined in Eq. (1). For the estimation of $\sigma_{Y,\mathrm{est}}$ by Eq. (17), the additional calculations of M_3 and M_4 are necessary.

B. Estimation Formula with consideration of only 2nd Order Neutron-correlation

Now, let us derive the approximation formula for the statistical error of Y value to be compared with that of the Poisson distribution, Eq. (14). In general, the 3rd and 4th order neutron-correlations are lower than the second order neutron-correlation, because the magnitude of *n*th order neutron correlation is proportional to the *n*th power of detector importance function, or detection efficiency [12]. Based on this approximation, approximation values of 3rd and 4th order factorial moments are derived as follows.

Firstly, a master equation for probability generating functions of neutron count is described as follows [7,12]:

$$\ln\big(G(Z,T|S)\big) = \int_0^\infty du \int_V dV\, S(\vec{r}) \sum_{q=0}^{\infty} p_s(q,\vec{r}) \times \left\{\big(\bar{g}(Z,T|\vec{r},u)\big)^q - 1\right\}, \tag{18}$$

$$G(Z,T|S) \equiv \sum_{C=0}^{\infty} Z^C P(C,T|S), \tag{19}$$

$$\bar{g}(Z,T|\vec{r},u) \equiv \int_0^\infty dE \int_{4\pi} d\Omega \frac{\chi_s(\vec{r},E)}{4\pi} g\big(Z,T|\vec{r},E,\vec{\Omega},u\big), \tag{20}$$

$$g\big(Z,T|\vec{r},E,\vec{\Omega},u\big) \equiv \sum_{C=0}^{\infty} Z^C p\big(C,T|\vec{r},E,\vec{\Omega},u\big), \tag{21}$$

where

- $p\big(C,T|\vec{r},E,\vec{\Omega},u\big)$: probability that C neutrons are detected during counting gate width T due to one neutron at $(\vec{r},E,\vec{\Omega},u)$, where u is a backward time variable, *i.e.* $u \equiv -t$, $u = 0$ corresponds to the counting gate closing time;
- $g\big(Z,T|\vec{r},E,\vec{\Omega},u\big)$: probability generating function for $p\big(C,T|\vec{r},E,\vec{\Omega},u\big)$;
- $\bar{g}(Z,T|\vec{r},u)$: weighted mean of probability generating function, of which weighting function is $\frac{\chi_s(\vec{r},E)}{4\pi}$;
- $P(C,T|S)$: probability that C neutrons are detected during counting gate width T due to stationary external neutron source S:
- $G(Z,T|S)$: probability generating function for $P(C,T|S)$;
- $S(\vec{r})$: spatial distribution of source strength;
- $p_s(q,\vec{r})$: probability that q neutrons are emitted per decay;
- $\chi_s(\vec{r},E)$: energy spectrum of external neutron source.

Using mathematical properties of probability generating function $G(Z,T|S)$ for Eq. (18), an nth order neutron-correlation value Y_n $(n \geq 2)$ is defined as:

$$Y_n = \frac{1}{\langle C\rangle} \frac{\partial^n \ln\big(G(Z,T|S)\big)}{\partial Z^n}\bigg|_{Z=1}, \tag{22}$$

where Y_2 corresponds to the Y value in the Feynman-α method. For example, 1st to 4th order partial derivatives of $\ln\big(G(Z,T|S)\big)$ with respect to Z are shown below:

$$\frac{\partial(\ln G)}{\partial z} = \frac{1}{G}\frac{\partial G}{\partial z}, \tag{23}$$

$$\frac{\partial^2(\ln G)}{\partial z^2} = \frac{1}{G}\frac{\partial^2 G}{\partial z^2} - \left(\frac{\partial(\ln G)}{\partial z}\right)^2, \tag{24}$$

$$\frac{\partial^3(\ln G)}{\partial z^3} = \frac{1}{G}\frac{\partial^3 G}{\partial z^3} - \left(\frac{\partial(\ln G)}{\partial z}\right)^3 - 3\left(\frac{\partial(\ln G)}{\partial z}\right)\frac{\partial^2(\ln G)}{\partial z^2}, \tag{25}$$

$$\frac{\partial^4(\ln G)}{\partial z^4} = \frac{1}{G}\frac{\partial^4 G}{\partial z^4} - \left(\frac{\partial(\ln G)}{\partial z}\right)^4 -6\left(\frac{\partial(\ln G)}{\partial z}\right)^2\frac{\partial^2(\ln G)}{\partial z^2} - 3\left(\frac{\partial^2(\ln G)}{\partial z^2}\right)^2 -4\frac{\partial(\ln G)}{\partial z}\frac{\partial^3(\ln G)}{\partial z^3}. \tag{26}$$

In addition, $G(Z,T|S)$ satisfies the following mathematical properties:

$$G(Z,T|S)|_{Z=1} = \sum_{C=0}^{\infty} P(C,T|S) = 1, \tag{27}$$

$$\frac{\partial^n G}{\partial Z^n}\bigg|_{Z=1} = \langle C(C-1)(C-2)\cdots(C-n+1)\rangle. \tag{28}$$

Using Eqs. (22)-(28), 2nd to 4th order neutron-correlation values can be expressed as:

$$Y \equiv Y_2 = \frac{\partial^2 \ln G}{\partial Z^2}\bigg|_{Z=1} = \frac{\langle C(C-1)\rangle - \langle C\rangle^2}{\langle C\rangle}, \tag{29}$$

$$Y_3 = \frac{\partial^3 \ln G}{\partial Z^3}\bigg|_{Z=1} = \frac{\langle C(C-1)(C-2)\rangle - \langle C\rangle^3 - 3Y\langle C\rangle^2}{\langle C\rangle}, \tag{30}$$

$$Y_4 = \frac{\partial^4\, ln\, G}{\partial Z^4}\bigg|_{Z=1} = \frac{1}{\langle C\rangle}\begin{pmatrix}\langle C(C-1)(C-2)(C-3)\rangle - \langle C\rangle^4 \\ -6Y\langle C\rangle^3 - 3Y^2\langle C\rangle^2 - 4Y_3\langle C\rangle^2\end{pmatrix}, \tag{31}$$

If Y_3 and Y_4 are negligible small in Eqs. (30) and (31), the 3rd and 4th order factorial moments are approximated by:

$$\langle C(C-1)(C-2)\rangle \approx \langle C\rangle^3 + 3Y\langle C\rangle^2, \tag{32}$$

$$\langle C(C-1)(C-2)(C-3)\rangle \approx \langle C\rangle^4 + 6Y\langle C\rangle^3 + 3Y^2\langle C\rangle^2. \tag{33}$$

By utilizing Eqs. (32) and (33) for the estimators of 3rd and 4th order central moments M_3 and M_4, the following approximation formulae can be derived:

$$M_3 \approx (1+3Y)\bar{C}, \tag{34}$$

$$M_4 \approx 3\{(1+Y)\bar{C}\}^2 + (1+7Y)\bar{C}. \tag{35}$$

By substituting Eqs. (34) and (35) into Eq. (17), the approximation formula for the statistical error $\sigma_{Y,\mathrm{2nd}}$ can be finally derived as follows:

$$\sigma_{Y,\mathrm{2nd}} \approx (1+Y)\sqrt{\frac{Y(1-Y)(2-Y)}{(1+Y)^2 N\bar{C}} + \frac{2}{N-1}}, \tag{36}$$

where Eq. (36) is applicable to the condition where the sign in the square-root of is positive, *e.g.* $Y < 1$. As compared Eq. (36) with Eq. (14), the statistical error σ_Y is corrected due to the 2nd order neutron correlation factor, Y. If the Feynman-α experiment is conducted under a situation where the total sum of neutron count $N\bar{C}$ is large enough, σ_Y can be mainly reduced by increasing N. In such a situation, a high strength external neutron source, or large $\bar{C}$, contributes little to improving the statistical error of Y, although the relative statistical error of mean $\langle\frac{\sigma_{\bar{C}}}{\bar{C}}\rangle$ can be reduced. It is noted that, the relative statistical error $\frac{\sigma_{Y,\mathrm{2nd}}}{Y}$ becomes smaller by increasing Y using a detector with higher efficiency, since the absolute value of Y is proportional to the detection efficiency. Hence, the improvement of the detection efficiency is important to reduce the relative error of Y value.

Using Eq. (36), the statistical error $\sigma_{Y,\mathrm{2nd}}$ can be approximately estimate by reusing Y value without calculation of M_3 and M_4, *i.e.* the calculations of $\bar{C}$ and s^2 are sufficient for the error estimation.

III. EXPERIMENTAL ANALYSIS

1. Experimental Condition

In the previous study, the reactor noise experiments were conducted in the A-core (A3/8"p36EU-NU) at the Kyoto University Critical Assembly (KUCA) [13]. The detail of this experiment is reported in the reference [9].

The experimental core and the loaded fuel assembly are shown in Fig. 1 and 2, respectively. In this experiment, ^{3}He detectors (#1~4) were placed at axially center positions of excore reflector assemblies. Using these detectors, the time-series data of neutron counts were successively measured. At the shutdown state, the reactor noise was measured without any external neutron source such as a Cf source, *i.e.* using only inherent neutron source which mainly consists of spontaneous fission of ^{238}U and (α,n) reactions of ^{27}Al due to α-decay of uranium isotopes [14]. The detector#2 is used for the present reactor noise analysis, where the neutron count rate $\bar{C}/T$ is 4.444 ± 0.011 [count/sec].

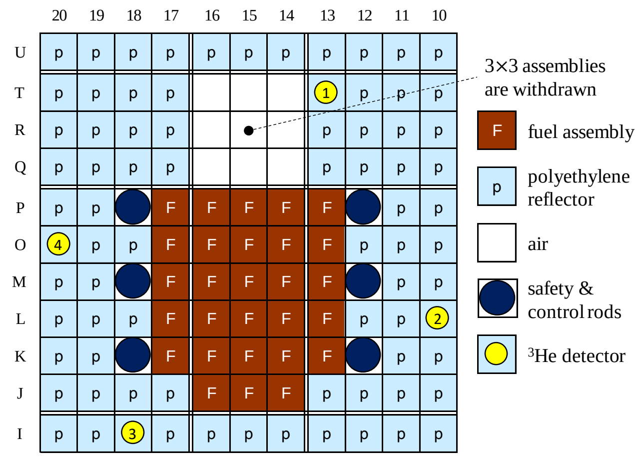

Fig. 1. Top view of experimental core (A3/8"p36EU-NU).

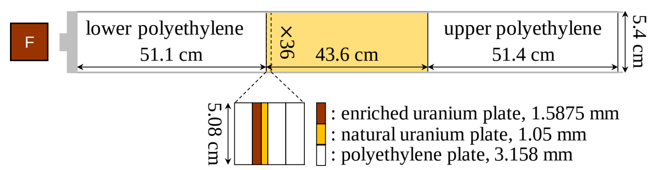

Fig. 2. Fuel assembly loaded in experimental core.

In order to measure the reference values of statistical error $\sigma_{Y,\mathrm{ref}}$, 93 times of 10 minutes' reactor noise measurements were conducted. Using the conventional bunching method [15], the variation of $Y(T)$ was individually evaluated for each of 10 minutes' measurements. When the counting gate width is T [sec], the number of counting gate N corresponds to $N = \lfloor 600/T \rfloor$. Using 93 sets of $Y_m(T)$, the reference statistical error $\sigma_{Y,\mathrm{ref}}$ were estimated.

$$\sigma_{Y,\mathrm{ref}} = \sqrt{\frac{1}{93-1}\sum_{m=1}^{93}\left\{Y_m - \left(\frac{1}{93}\sum_{m'=1}^{93} Y_{m'}\right)\right\}^2}, \tag{37}$$

In order to confirm the validity of error estimation formulae, one of the 10 minutes-reactor noise measurements is selected. Then, the statistical errors $\sigma_{Y,\mathrm{est}}$ and $\sigma_{Y,\mathrm{2nd}}$ are estimated by both Eqs. (17) and (36). For discussion about the 2nd order neutron correlation effect, the approximated statistical error $\sigma_{Y,\mathrm{P}}$ using the Poisson distribution is also evaluated using Eq. (14).

Furthermore, as an alternative error estimation technique, the bootstrap standard deviation σ_{Y^*} is numerically evaluated by the bootstrap method. In the bootstrap method, a histogram of original neutron count data is utilized as an experimentally-based probability distribution in the resampling to evaluate the statistical error of Y, *e.g.* the bootstrap-standard deviation σ_{Y^*} and the bootstrap-confidence interval. The detail of this methodology is explained in reference [9]. In Appendix A, the brief explanation about the estimation of σ_{Y^*} is summarized.

2. Results

Figure 3 shows statistical errors of (1) reference $\sigma_{Y,\mathrm{ref}}$, (2) approximated $\sigma_{Y,\mathrm{P}}$ using the Poisson distribution, (3) $\sigma_{Y,\mathrm{est}}$ using the unbiased estimators for 3rd and 4th central moments, (4) $\sigma_{Y,\mathrm{2nd}}$ with consideration of only 2nd order neutron-correlation, and (5) σ_{Y^*} by the bootstrap method.

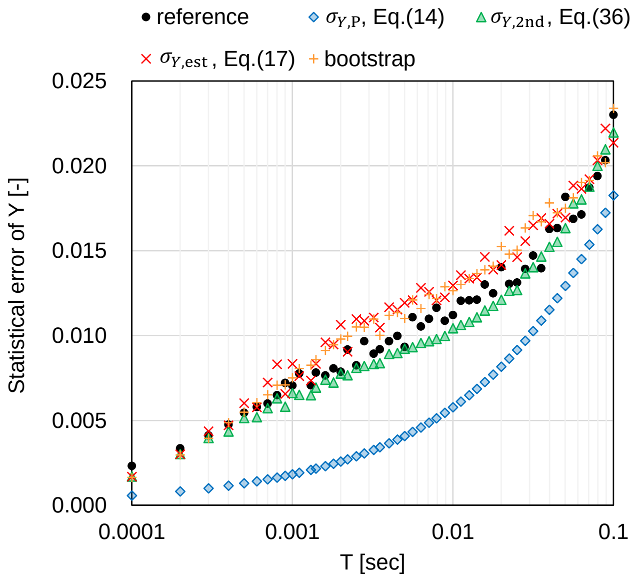

Fig. 3. Estimation results of statistical error of Y value.

As shown in Fig. 3, $\sigma_{Y,\mathrm{est}}$, $\sigma_{Y,\mathrm{2nd}}$, and σ_{Y^*} agree well with reference $\sigma_{Y,\mathrm{ref}}$. Compared with these results, $\sigma_{Y,\mathrm{P}}$ is underestimated, although $\sigma_{Y,\mathrm{P}}$ is the simplest way to roughly guess the statistical error without measured neutron count data, *i.e.* Eq. (14) needs only N which can be given beforehand in the experimental design stage.

By comparison between $\sigma_{Y,\mathrm{P}}$ and $\sigma_{Y,\mathrm{2nd}}$, it is confirmed that the 2nd order neutron-correlation effect is important to improve the estimation of the statistical error σ_Y. From the fact that $\sigma_{Y,\mathrm{2nd}}$ is nearly equal to or slightly smaller than $\sigma_{Y,\mathrm{est}}$, it is demonstrated that the 3rd and 4th order neutron-correlations have small impact on the estimation of σ_Y. The advantage of $\sigma_{Y,\mathrm{2nd}}$ is lower calculation cost, *i.e.* the statistical error can be obtained from the measurement values $\bar{C}$ and Y only. Although $\sigma_{Y,\mathrm{est}}$ needs additional calculation for M_3 and M_4, but the calculation cost is insignificant. Thus $\sigma_{Y,\mathrm{est}}$ is also one of the practical estimation methods.

As previously reported in the reference [9], the bootstrap method can provide the reasonable statistical error such as the bootstrap standard deviation σ_{Y^*}. As shown in Fig. 3, σ_{Y^*} and $\sigma_{Y,\mathrm{est}}$ are almost the same. The disadvantage of bootstrap method is that calculation cost is relatively high due to the resampling procedures. In the present analysis, the bootstrap replicates Y^* were randomly resampled 1000 times to obtain σ_{Y^*} with high precision, thus the total calculation time of the bootstrap method is approximately at least 1000 times higher than that of $\sigma_{Y,\mathrm{2nd}}$ and $\sigma_{Y,\mathrm{est}}$. Because of this calculation time, the bootstrap method may be unsuitable for the real-time statistical error estimation in the on-line monitoring system. Note, however, that the bootstrap method enables us to easily estimate not only the statistical error of Y but also the statistical error of prompt neutron decay constant α (denoted as σ_α). In the present study, the authors have only derived the theoretical formulae for σ_Y. The theoretical derivation for σ_α is a one of the future tasks, because the derivation of σ_α is more complicated due to the fitting procedure to evaluate α.

Consequently, it is validated that the derived theoretical formulae, Eqs. (17) and (36), are useful to estimate the statistical error of Y value from a single measurement of reactor noise. By utilizing these estimation formulae, the variation of $Y(T)$ can be measured with the statistical error, as shown in Fig. 4. The estimated statistical error is also applicable to the weight in the fitting procedure for α.

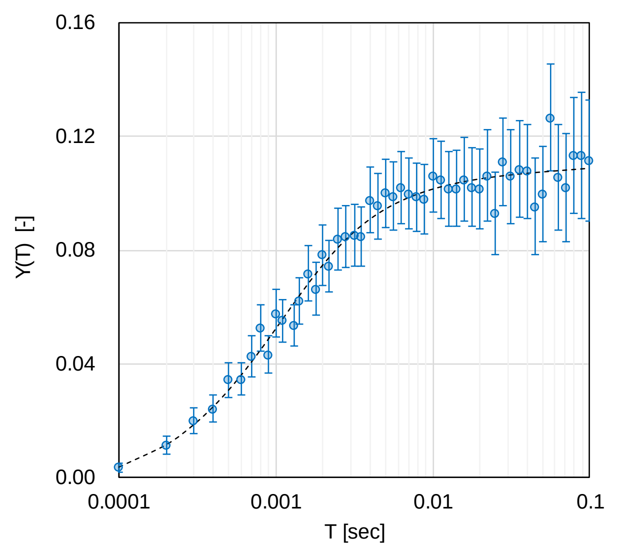

Fig. 4. Variation of $Y(T)$ with the statistical error $\sigma_{Y,\mathrm{est}}$ (error bar represents 1σ).

IV. CONCLUSION

As the statistical error estimation for Y value from a single measurement of reactor noise, the practical estimation formulae of $\sigma_{Y,\mathrm{est}}$ and $\sigma_{Y,\mathrm{2nd}}$, or Eqs. (17) and (36), were newly derived. The derived formulae clarified that the statistical error σ_Y can be reduced by the total number of counting gate N (or total measurement time NT) rather than the strength of external neutron source. It is noted that the relative statistical error $\frac{\sigma_Y}{Y}$ can be improved using a detector with higher efficiency.

Through the reactor noise analysis for the actual KUCA experiment, it was validated that the statistical errors $\sigma_{Y,\mathrm{est}}$

and $\sigma_{Y,\mathrm{2nd}}$ using Eqs. (17) and (36) agree well with the reference value of $\sigma_{Y,\mathrm{ref}}$ which was obtained from multiple measurements of reactor noise. Furthermore, it was confirmed that the 2$^{\mathrm{nd}}$ order neutron-correlation effect is important in the estimation of the statistical error of Y value by comparing with the approximated statistical error $\sigma_{Y,\mathrm{P}}$ using the Poisson distribution. The estimated error of $\sigma_{Y,\mathrm{est}}$ using the unbiased estimators for 3$^{\mathrm{rd}}$ and 4$^{\mathrm{th}}$ central moments was approximately equal to that of the bootstrap method. Compared with the bootstrap method, the advantage of the practical estimation formulae of $\sigma_{Y,\mathrm{est}}$ and $\sigma_{Y,\mathrm{2nd}}$ is lower calculation cost.

APPENDIX A: BOOTSTRAP METHOD

The estimation procedures of statistical error σ_{Y^*} are briefly explained below:

1. Original time-series data of neutron counts $C_i(T_0)$ are provided by a single measurement of reactor noise, where T_0 is a basic counting gate width. The total number of count data is N_0.
2. Set k be an arbitrary number of bunching $(1 \le k < N_0)$.
3. The "resampling position r" is determined using a uniform random integer number, $1 \le r \le (N_0 - k + 1)$. Then, neutron count $C^*(kT_0)$ is resampled by bunching the successive count data as follows:
$$C^*(kT_0) = \sum_{i=r}^{r+k-1} C_i(T_0). \tag{A.1}$$
4. By repeating $K(= \lfloor N_0/k \rfloor)$ times of random-resampling described in step 3, then "bootstrap sample" of count data is newly generated as follows:
$$\vec{C}^*(kT_0) \equiv \{C_1^*(kT_0), C_2^*(kT_0), \cdots, C_K^*(kT_0)\}. \tag{A.2}$$
5. Using Eq. (1) for $\vec{C}^*(kT_0)$, "bootstrap replicate $Y^*(kT_0)$" is evaluated for the bunching gate width kT_0.
6. Repeat steps 2 through 5 by varying k to obtain the variation of Y^* with respect to counting gate width.
7. In order to estimate standard deviation of the bootstrap replicate Y^*, repeat steps 2 through 6 several times. Consequently, many number of bootstrap replicates Y^{*b} are obtained for $b = 1,2,\cdots,B$. Here, B is the number of bootstrap replicates.
8. Using dataset of Y^{*b} in step 7, the bootstrap standard deviation of Y^* (denoted as σ_{Y^*}) is calculated for each counting gate width kT_0:

$$\sigma_{Y^*} = \sqrt{\frac{1}{B-1}\sum_{b=1}^{B}\left(Y^{*b} - \frac{1}{B}\sum_{b'=1}^{B} Y^{*b'}\right)^2}. \tag{A.3}$$

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