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Some New Thoughts on the Multipoint Method for Reactor Physics Applications

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Abstract - *The multipoint approach proves to be an efficient method for the neutron kinetics of nuclear reactors and accelerator-drive systems. In the present work some features of multipoint kinetics are discussed with reference to standard discretization schemes, both in space and in energy. In the second part of the paper a physically-base approach to construct a multipoint model is presented. It is shown how the parameters of the model can be derived through a Monte Carlo simulation evaluating transfer probabilities and characteristic times. Some results are obtained for an idealized fast system to evidence the properties of the multipoint parameters. At last, a standard point kinetic model is reconstructed from the multipoint equations, showing the relationship between the classic integral parameters of a multiplying systems and the multipoint parameters. This allows to consider a multipoint model for inverse applications aiming at the experimental determination of integral parameters.*

I. INTRODUCTION

In the study of nuclear reactors it is of great importance to accurately simulate the spatial and spectral effects during transient situations. In many applications only the very simple point kinetic model is used. This model is particularly advantageous when dealing with inverse problems, e.g. for the interpretation of kinetic experiments to reconstruct integral parameters such as reactivity or effective delayed neutron fractions, from flux measurements [1]. However, point kinetics is not suitable at all for applications to large reactors or uncoupled systems characterized by large spatial distortions during transients. It is also well-known that point-kinetic results may be on the unsafety side, as the values of the power may turn out to be lower than the exact ones. A full space-energy approach may be computationally too intensive in many applications. Alternatively, the multipoint approach can prove to yield adequate results in many applications with an acceptable computational effort.

The idea of the multipoint method is based on subdividing the reactor phase space (geometrical domain and/or energy range) into separated subdomains and assuming that the characteristic function (amplitude) for each subdomain evolves according to a point-like equation, including coupling terms with the other subdomains, due to neutron streaming in space and transfer in energy through collisions. Therefore a system of coupled first-order ordinary differential equations is obtained.

The multipoint model was proposed by Avery in the theory of coupled reactors [2, 3]. Later several formulations were proposed in different frameworks [4, 5, 6, 7].

In this paper some basic aspects of the multipoint formulation are considered, starting from simple physical configurations. The attention is focused separately on spatial and spectral effects. For spatial problem the criticality problem is analysed, giving some attention also to the full eigenvalue spectrum of the multipoint equations.

One attractive feature of the multipoint method is the possibility of its application to inverse problems for the interpretation of kinetic experiments, in particular for the measurement of integral parameters in source-driven experiments carried out in subritical assemblies for the assessment of accelerator-driven technology. In many situations point kinetics has proved to yield unsatisfactory results [8]. In this work the point kinetic model is reformulated starting from the multipoint one, in order to relate the standard integral parameters of point kinetics to the multipoint parameters.

A physical-based general approach for the formulation of the multipoint equations is thoroughly discussed. The parameters are related to probabilities associated to the transport and collision phenomena within the system. Such parameters are evaluated using a Monte Carlo approach: results for a challenging fast system configuration are presented.

The general features of the method are described in the following without directly introducing the source term. However, the contribution of an external source can be readily included. Therefore, the method is applicable for the description and the simulation of an accelerator-driven system.

II. SPACE MULTIPOINT THEORY

The multipoint method can be rigorously derived through a standard separation-projection procedure, generalizing the Henry's process to derive the point kinetic equations [7]. We are using here formulations based on a direct approach. If one focuses on space aspects only, a multipoint set of equations can be obtained by direct discretization of the spatial variable by means of a standard discretization scheme. Of course, this is not what is done in effective multipoint models, because standard discretization schemes require to set the spatial mesh to values that can effectively account for the physical process of neutron motion. Therefore, due to the values of cross sections in realistic systems, the number of points may turn out to be too large and far from matching the philosophy of the multipoint approach. This is the reason why more sophisticated procedures must be used, such as the already mentioned separation-projection technique or the physical here proposed. However, the use of spatially discretized equations

may prove useful to understand the numerical features of the multipoint framework.

1. The space discretized model

Let us refer to a simple one-group problem, let us also define a (n+1)-point spatial grid and a vector $\underline{\phi}$ containing the values of the scalar fluxes at each point of the grid. The discretized balance equations in the absence of a neutron source take the form:

$$\frac{1}{\mathrm{v}}\frac{d}{dt}\underline{\phi}(t) = \hat{F}\underline{\phi}(t) - \hat{L}\underline{\phi}, \tag{1}$$

where the loss matrix $\hat{L}$ and the fission matrix $\hat{F}$ are related to the material properties of the domain, the physical model adopted and the discretization scheme. For instance, in a simple homogeneous one-dimensional plane medium of width l, with a uniform grid characterized by the mesh length h and using the diffusion model while disregarding delayed emissions, the loss matrix $\hat{L}$ is tridiagonal, with $l_{ii} = 2\frac{D}{h^2} + \Sigma_a$ and $l_{i,i+1} = l_{i,i-1} = -\frac{D}{h^2}$, while $\hat{F}$ is a diagonal matrix whose elements are simply equal to $\nu\Sigma_f$. In this case the matrices have dimensions n-1 because boundary conditions impose $\phi_0 = \phi_n = 0$. A criticality theory may be developed from the above model, by studying the steady-state version of Eq. (1) and introducing a coefficient $1/k$ in front of the fission term, thus moving to an eigenvalue problem. Matrix $\hat{L}$ is a very peculiar one, that is a tridiagonal Toeplitz matrix, characterized by constant elements on each of the descending diagonals, for which an analytical formula for the eigenvalues is available [9], leading, with some algebraic manipulations, to an expression for the spectrum of the k eigenvalues

$$k_m = \frac{\nu\Sigma_f}{\frac{2Dn^2}{l^2}\left(1 + \cos\left(\frac{m\pi}{n}\right)\right) + \Sigma_a}, \qquad m = 1, \ldots, n-1 \tag{2}$$

for an n interval mesh. In the limit $n \to \infty$, using

$$\lim_{n\to\infty} n^2\left[1 + \cos\left(\frac{(n-q)\pi}{n}\right)\right] = \frac{\pi^2 q^2}{2},$$

we obtain the spectrum for the k eigenvalues as

$$k_q = \frac{\nu\Sigma_f}{D\frac{\pi^2 q^2}{l^2} + \Sigma_a}, \qquad q = 1, \ldots, \infty, \tag{3}$$

where we use $q = n - m$, so that now k_1, the highest eigenvalue can be truly interpreted as k. In Fig. 1 we show the spectrum of the eigenvalues as a function of the number of intervals in the mesh: we represent all the eigenvalues for a given mesh dimension with the same color; it is interesting to observe that all eigenvalues are real and that the m^{th} eigenvalue, whose first appearance is when the number of intervals is equal to $m + 1$, is really an approximation to the m^{th} k eigenvalue, to which it converges monotonically as the number of meshes increases.

For a non-homogeneous medium the same formulation of the problem applies as well, provided one replaces the constant values $D, \Sigma_a, \nu\Sigma_f$ with the corresponding values evaluated at the mesh points $D_j = D\left(x_j\right), \Sigma_{a,j} = \Sigma_a\left(x_j\right), (\nu\Sigma_f)_j = \nu\Sigma_f\left(x_j\right)$; in this case, however, the Toeplitz nature of the tridiagonal matrix $\hat{L} - \frac{1}{k}\hat{F}$ is lost and the analytical form for the spectrum is no more available, but a numerical evaluation of the required determinant remains feasible.

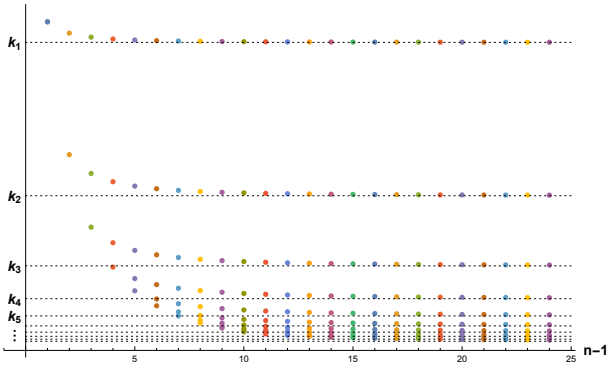

Fig. 1. The k eigenvalue spectrum for a one-dimensional homogeneous slab as a function of the number of points in the spatial mesh.

2. A physical approach to multipoint model

A general approach is possible, grounded on a physical basis: we consider a reactor as subdivided into a set of non overlapping geometrical parts, whose union covers the full domain, and we try to build a proper balance model. Let us start for simplicity with a 2-zone system, and ask how the number of neutrons born in each zone - which at time t are everywhere in the system - can vary over time. The balance equations in the absence of an external source, that can be written almost immediately, read:

$$\begin{cases} \dfrac{dN_1}{dt} = \nu\left(a_{11}N_1 + a_{12}N_2\right) - C_1N_1 \\ \dfrac{dN_2}{dt} = \nu\left(a_{21}N_1 + a_{22}N_2\right) - C_2N_2\,. \end{cases} \tag{4}$$

A neutron is generated - in this scheme - only via a fission process. Hence, for instance, a fission can happen in zone 1 generating ν new neutrons, but loosing the original one: this event can be induced by a neutron born in zone 1 with a probability per unit time a_{11} or by a neutron originated in zone 2 with a probability per unit time a_{12}; the same processes can happen in zone 2, but there is also the possibility that a neutron in zone 1(2) is absorbed (or leak out of the system) with probabilities per unit time $C_{1(2)}$.

In order to be more explicit, for instance, the quantity $a_{12}N_2$ is the contribution of neutrons generated in zone 2 that produces a fission zone 1 at time t and thence are in zone 1 at time t itself. To write down these equations we considered solely two physical processes to happen, in both zones: the production of ν neutrons by a fission process and the absorption of neutrons in each of the reactor zones; this approach implies that the material properties of the system are assumed to be stationary. If, as in many realistic cases,

the material properties are changed, either because of control operations, or accidental events or feedback phenomena, the parameters must be updated during the transient. If this is the case, an approximation is implicit, to neglect the effect of the time delay needed to transport neutrons from the zone where it was generated by a fission process to the zone where finally it will be absorbed, giving origin or not to a new fission process. However the time scale associated to the neutron transport is much faster than any other physical phenomena involved in the changes of the material properties of the system.

Naturally the quantities N_i are by themselves not directly measurable. On the other hand the quantity $a_{11}N_1 + a_{12}N_2$ being the total fission rate in zone is, in principle, an experimental observable.

Within this scheme a consistency requirement is implicitly assumed: namely, the subdivision of the system in (spatial) zones must be performed in such a way that each one of them contains fissile materials. This is consistent with the assumed philosophy underlying lumped models [2]. It must be observed that the choice of the subdivision of the spatial domain is not unique: the model obtained must be anyway assessed on physical grounds and validated to assure the quality of the results. For instance, a quite natural suggestion could be to "equalize" the values of the total content of the neutron importance - that is the integral over a zone of the product of the importance and the neutron flux - in each of the zones.

The coefficients just introduced define coupling matrices for the system. This approach recalls, in many aspects, the collision probability method used in integral neutron transport [10]. Equations (4) can be written in matrix form - so to be easily generalizable to a M zone reactor - as:

$$\frac{d}{dt}\underline{N}(t) = \nu\hat{A}\underline{N}(t) - \hat{C}\underline{N}(t), \tag{5}$$

where $\hat{C}$ is a diagonal matrix with elements $\left(\hat{C}\right)_{ii} = C_i$ and obviously $\left(\hat{A}\right)_{ij} = a_{ij}$. An eigenvalue equation can be generated by considering an artificial time independent system, where only ν/k neutrons are generated by fission. The multiplication eigenvalue k is then determined by:

$$\det\left[\frac{\nu}{k}\hat{A} - \hat{C}\right] = 0. \tag{6}$$

Again we have M values for k, but in principle they can be real or even complex: however we know by the previous example that in the large M limit they must become real, approximating the true spectrum of k values, at least when adopting the diffusion model.

The inclusion of delayed neutrons does not present particular difficulties:

$$\begin{cases} \dfrac{d\underline{N}}{dt} = \nu(1-\beta)\hat{A}\cdot\underline{N} - \hat{C}\cdot\underline{N} + \displaystyle\sum_{\alpha=1}^{N_d}\lambda_\alpha\underline{\mathbb{C}}^{(\alpha)} \\ \dfrac{d\underline{\mathbb{C}}^{(\alpha)}}{dt} = \beta_\alpha\nu\hat{A}\cdot\underline{N} - \lambda_\alpha\underline{\mathbb{C}}^{(\alpha)}, \end{cases} \tag{7}$$

with the same meaning as before for the matrices $\hat{A}$, $\hat{C}$; here $\beta = \sum_\alpha^{N_d}\beta_\alpha$; we indicate with $\underline{\mathbb{C}}^{(k)}$ the concentrations of precursors for the k^{th} family of delayed neutrons in all the zones of the reactor, with decay constant λ_α, respectively.

This heuristic approach to space multipoint kinetics generates a richer physics than the naïve one obtained by space discretization, because, in general, all zones are coupled, not only near neighbours.

In a realistic situation, the evaluation of the coupling matrices can be carried out effectively by the Monte Carlo method. This approach is particularly viable in the case the properties of the system are stationary, such as for subcritical experiments, where the transient is caused by neutron source variations. In the case of transients induced by material changes, the matrices should be evaluated along the evolution to account for such changes, as previously pointed out.

III. SOME RESULTS

In this Section we present the values of the multipoint matrices obtained by Monte Carlo evaluation of the required rates. The evaluations have been carried out using the code MCNP6 [11]; as a first step we estimated the probabilities that neutrons born uniformly in zone j ultimately fission in zone i or that neutrons from zone j are absorbed (or leak) anywhere in the system in steady state conditions. The source neutrons are assumed spatially uniformly distributed, which of course is an approximation that looks reasonable in a fast reactor system as the one under consideration. As a result we obtain the probability matrix $\{p_{ij}\}$ and the probability absorption vector $\{c_j\}$.

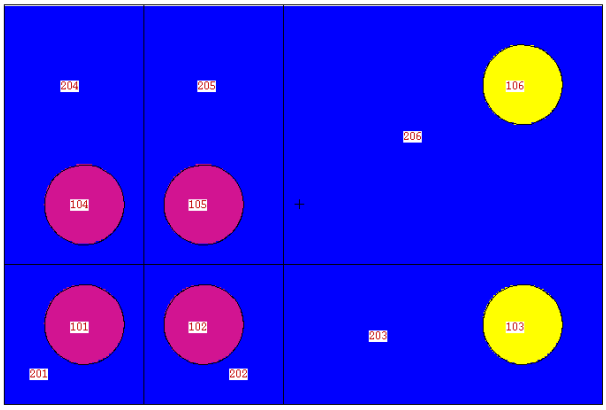

Fig. 2. Geometry used for the test case. Fuel regions are shown in red and yellow, lead ones in blue, see text for details. This plot was obtained by the model in MCNP6 and the numbers identify different MCNP6 cells. For the present approach we identify zone 1 with the union of cells 101 and 201, zone 2 with the union of cells 102 and 202, and so on. The problem is bi-dimensional, having used reflecting surfaces as bounds along the z direction.

As a second step the prompt neutron generation times associated to neutrons generated in zone j, Λ_j, are estimated as the average time value from the birth of a neutron to its death. This was done by tallying the times at which each neutron born in zone j is absorbed anywhere in the system

and evaluating the mean. As a consequence we set

$$a_{ij} = \frac{p_{ij}}{\Lambda_j}\,, \qquad C_j = \frac{c_j}{\Lambda_j}\,, \tag{8}$$

which are the multipoint parameters needed in model (7).

A comment concerning the Monte Carlo simulations is here in order. Unlike what has to be done for normal simulations, here every neutron history is terminated whenever the neutron either is absorbed or leaks out of the system, without the need to simulate the secondary neutrons emitted by fission. In fact, recalling the previous definitions of the multipoint parameters, only the information on the birth and on the death of the neutron is needed. As a consequence, the present method can be applied to any type of system, including even supercritical reactors. Furthermore, the computational burden is reduced with respect to standard simulations and it does not depend on the value of the multiplication factor.

We used a physically challenging non symmetric heterogeneous geometry, shown in Figure 2, to better investigate the implications of zone subdivisions. The system is a solid lead parallelepiped of $150\times100\times100$ cm^3 volume with cylindrical fuel insertions of 10 cm radius. The problem is made effectively bi-dimensional by using reflective bounding surfaces in the z direction.

We used lead in the blue areas, while fuel is a mixture of oxygen and uranium. The ^{235}U atomic fraction was 25% in cells 103 and 106, while in the other four was chosen so to fix k_{eff}, ranging from $\sim$ 22.6% for $k_{eff} = 0.94$ to $\sim$ 26.6% for $k_{eff} = 1$.

k_{eff}	ν	k_2	k_3	k_4	k_5	k_6
0.99	2.61811	0.779	0.743	0.731	0.667	0.561
0.98	2.61847	0.778	0.735	0.731	0.659	0.555
0.97	2.61871	0.778	0.732	0.726	0.652	0.548
0.96	2.61891	0.777	0.732	0.718	0.644	0.542
0.95	2.61914	0.777	0.732	0.710	0.637	0.535
0.94	2.61934	0.777	0.732	0.702	0.630	0.529

TABLE IV. Higher order k-eigenvalues. ν values have been obtained by fixing the highest multiplication eigenvalue to the k_{eff} value obtained from simulation.

	$k_{eff} = 0.99$		$k_{eff} = 0.94$	
zone (z)	Λ_z [μs]	$\Lambda_{z,leak}$ [μs]	Λ_z [μs]	$\Lambda_{z,leak}$ [μs]
1	1.29	1.09	1.41	1.15
2	2.25	2.98	2.45	3.13
3	2.52	2.48	2.53	2.49
4	1.73	2.21	1.89	2.35
5	3.16	7.45	3.42	7.85
6	2.52	2.53	2.53	2.54

TABLE V. Values of the characteristic mean absorption times Λ_z and mean leakage times $\Lambda_{z,leak}$ for neutrons generated in all zones and for two values of k_{eff}.

The probability matrices and absorption vectors for two values of k_{eff} are presented in Table I, II, where also fission and leakage probabilities are reported. It can be clearly seen that the probability matrix are largely dominated by diagonal elements. Furthermore, one can observe the effect of the subcriticality is apparently not very strong, although such slight differences produce a large reactivity change: as for other reactor physics problems the parameters of the model have to be accurately evaluated. For the present results the Monte Carlo accuracies obtained are below 0.1%. Table III collects the values of the mean neutron absorption times corresponding to all the zones of the system for different values of k_{eff}.

When considering systems characterized by low-absorption non fuel materials, such as generally is for fast systems and for the case here considered, the choice of geometrical splitting of the spatial domain into the multipoint structure is marginal because the contribution to the rates from non fuel portions of the zones is consequently low. For instance, moving the boundary between region 2, 5 and 4, 6 by 50 cm to the right, it has been verified that the probability matrix and absorption vector are slightly affected.

We now present some results for the spectrum at various k_{eff}: however we do not have any way to know "a priori" the precise value for ν to be used for the problem. Then a first possibility is to fix the highest eigenvalue from (6) to the corresponding k_{eff} value and use it for the evaluation of ν to be used to extract higher order k eigenvalues. Results are presented in Table IV. The analysis of higher order eigenvalues is of interest to predict the potential presence of spatial effects in time-dependent configurations [12].

As it can be immediately realized from the second colums of Table IV there is a minimal increment of ν as k_{eff} decreases: in fact, to decrease k_{eff} we decrease the ^{235}U fraction, as it was explained before; correspondingly the ^{238}U fraction increases, yielding a very small increment on the effective value of ν due to fast fission in the incident neutron energy range above 1 MeV, where the fission cross section becomes relevant and ν is sharply increasing.

It must be observed that the use of an effective value of ν, determined in order to preserve the reference value of k_{eff} from the Monte Carlo simulation, takes into account and compensates in an overall fashion for all the effects associated to the different effective fission, absorption and leakage proper times for each zone. Formula (8) assumes equal characteristic times for all phenomena, while, of course, physically they may be quite different. In particular, the leakage times may be significantly different, depending on the localization of the zone under analysis, as can be clearly seen in Table V, where the values adopted in formula (8) are compared to the mean leakage times. As a last observation, the average fission times are expected to be shorter than absorption times, because most of the fission events happen in the same fuel region as the one of the neutron birth.

An alternative more consistent procedure to avoid the intermediate estimate of the number of secondary neutrons emitted by fission, would consist in directly estimating the rate of fission-emitted neutrons, thus obtaining the values of the probabilities $\tilde{p}_{ij}$, so allowing to directly evaluate the elements of $\nu\hat{A}$ as:

$$\nu a_{ij} = \frac{\tilde{p}_{ij}}{\tilde{\Lambda}_j}\,, \tag{9}$$

Zone (z)	P_{leak}	P_{capt}	P_{fiss}	p_{z1}	p_{z2}	p_{z3}	p_{z4}	p_{z5}	p_{z6}
1	0.3356	0.3081	0.3563	0.2995	$3.32\ 10^{-2}$	$2.29\ 10^{-4}$	$1.48\ 10^{-2}$	$8.36\ 10^{-3}$	$1.81\ 10^{-4}$
2	0.2683	0.3444	0.3872	$3.30\ 10^{-2}$	0.3167	$3.60\ 10^{-3}$	$8.37\ 10^{-3}$	$2.32\ 10^{-2}$	$2.25\ 10^{-3}$
3	0.4176	0.2767	0.3057	$2.36\ 10^{-4}$	$4.67\ 10^{-3}$	0.2880	$3.34\ 10^{-4}$	$3.75\ 10^{-3}$	$8.66\ 10^{-3}$
4	0.2853	0.3348	0.3798	$7.91\ 10^{-2}$	$2.09\ 10^{-2}$	$4.96\ 10^{-4}$	0.2478	$3.10\ 10^{-2}$	$6.16\ 10^{-4}$
5	0.1875	0.3884	0.4241	$2.04\ 10^{-2}$	$9.56\ 10^{-2}$	$4.88\ 10^{-3}$	$3.07\ 10^{-2}$	0.2675	$5.10\ 10^{-2}$
6	0.4195	0.2758	0.3046	$1.85\ 10^{-4}$	$3.08\ 10^{-3}$	$8.65\ 10^{-3}$	$4.57\ 10^{-4}$	$4.22\ 10^{-3}$	0.2881

TABLE I. Probabilities for leakage (P_{leak}), capture (P_{capt}), fission (P_{fiss}) and zone-to-zone transmission for $k_{eff} = 0.99$.

Zone (z)	P_{leak}	P_{capt}	P_{fiss}	p_{z1}	p_{z2}	p_{z3}	p_{z4}	p_{z5}	p_{z6}
1	0.3408	0.3217	0.3375	0.2827	$3.19\ 10^{-2}$	$2.36\ 10^{-4}$	$1.44\ 10^{-2}$	$8.12\ 10^{-3}$	$1.87\ 10^{-4}$
2	0.2726	0.3597	0.3677	$3.18\ 10^{-2}$	0.2993	$3.66\ 10^{-3}$	$8.12\ 10^{-3}$	$2.25\ 10^{-2}$	$2.30\ 10^{-3}$
3	0.4177	0.2770	0.3052	$2.30\ 10^{-4}$	$4.43\ 10^{-3}$	0.2803	$3.22\ 10^{-4}$	$3.57\ 10^{-3}$	$8.66\ 10^{-3}$
4	0.2899	0.3498	0.3603	$7.51\ 10^{-2}$	$2.01\ 10^{-2}$	$5.09\ 10^{-4}$	0.2340	$2.99\ 10^{-2}$	$6.32\ 10^{-4}$
5	0.1907	0.4057	0.4035	$1.96\ 10^{-2}$	$9.10\ 10^{-2}$	$4.97\ 10^{-3}$	$2.96\ 10^{-2}$	0.2531	$5.20\ 10^{-2}$
6	0.4196	0.2761	0.3043	$1.81\ 10^{-4}$	$2.92\ 10^{-3}$	$8.65\ 10^{-3}$	$4.40\ 10^{-4}$	$4.01\ 10^{-3}$	0.2881

TABLE II. Probabilities for leakage (P_{leak}), capture (P_{capt}), fission (P_{fiss}) and zone-to-zone transmission for $k_{eff} = 0.94$.

where $\tilde{\Lambda}_j$ is the characteristic mean fission time.

IV. ENERGY MULTIPOINT THEORY

The multigroup equations can already be considered as a multipoint model on the energy variable: there are however some modifications to be introduced, coming from the fission or delayed spectrum: of the ν fission neutrons produced only a fraction χ_j is really produced in the j^{th} energy group:

$$\begin{cases} \dfrac{dN_i}{dt} = \nu(1-\beta)\chi_i \displaystyle\sum_j a_{ij}N_j - \\ \qquad\qquad - C_iN_i + \displaystyle\sum_{\alpha=1}^{N_d} \chi_i^{(\alpha)}\lambda_\alpha \mathbb{C}^{(k)} \\ \dfrac{d\mathbb{C}^{(\alpha)}}{dt} = \beta_\alpha \nu \displaystyle\sum_{ij} a_{ij}N_j - \lambda_\alpha \mathbb{C}^{(\alpha)}, \end{cases} \tag{10}$$

where $i = 1, \ldots, N_G, \alpha = 1, \ldots, N_d$ and $\chi_i(\chi_i^{(\alpha)})$ represents the probability that a prompt (delayed from α^{th} family) fission neutron has its energy in the i^{th} energy group.

It is worthwhile to remark that in the space multipoint approach both prompt neutron numbers and delayed neutron precursor numbers $\underline{N}, \underline{\mathbb{C}}$ are defined in the zone space, while in energy multipoint the latter are clearly scalars. This fact, as we shall see, requires some attention when building the general multipoint kinetic equations.

In ref. [13] we studied in detail the multiplication eigenvalue spectrum within a multigroup diffusion theory. In particular we proved a general expression for the eigenvalue k_n which is associated with the n^{th} spatial mode of the system which turns out to be necessarily real: the proof derived essentially from the peculiar form of the fission matrix - in that case of rank 1.

This feature is in principle lost in the general approach outlined in the static version of Eq. (10), because no general assumptions can be made on the structure of the matrix $\hat{A}$, so allowing again a richer physics: this in strict analogy to what we noticed in the discussion on Eq. (6) and (7).

In the previous Section we briefly discussed the dependence of the spectrum of k as a function of the number of space intervals (see Fig. 1). Analogously, it is possible to study the dependence of k on the number of energy groups in a naïve multigroup approach for an infinite system: however it can be shown that if one collapses macroscopic cross sections preserving reaction rates, the value of k remains unchanged.

V. REACTOR INTEGRAL PARAMETERS FROM THE MULTIPOINT MODEL

An important issue is the determination of integral parameters from local flux measurements. When using the simple point kinetic model, spatial effects may cause large errors in the prediction of integral parameters. Correction factors or flux-signal combination techniques should be adopted to obtain reliable values. The use of a multipoint model may prove to be more effective.

It can be assumed that the system is represented by a multipoint set of equations, where each point is associated to a local flux detector. The response of a source-driven system is described by a set of equations that takes the general form as in Eq. (7), that includes also the contribution of delayed neutrons, with an additional source term (a vector in the space indexed by zones).

The application of an inverse approach to an experimental measurement allows to estimate the elements of the characteristic matrices of the model. The solution of an adjoint problem is then used to project the above equations in order to retrieve a point-like set of equations, in the frame of the well-known Henry's philosophy.

Zone (z)	$k_{eff} = 0.99$	$k_{eff} = 0.98$	$k_{eff} = 0.97$	$k_{eff} = 0.96$	$k_{eff} = 0.95$	$k_{eff} = 0.94$
1	1.29	1.31	1.34	1.36	1.39	1.41
2	2.25	2.29	2.33	2.37	2.41	2.45
3	2.52	2.52	2.52	2.53	2.53	2.53
4	1.73	1.76	1.80	1.82	1.86	1.89
5	3.16	3.11	3.26	3.31	3.37	3.42
6	2.52	2.52	2.53	2.53	2.53	2.53

TABLE III. Mean neutron absoprtion times Λ_j $[\mu s]$ for neutrons generated in the 6 zones of the system, for different values of k_{eff}.

In the case of a space multipoint theory we introduce the direct eigenvector of the matrix $\hat{M} = (\nu - 1)(1 - \beta)\hat{A} - \hat{C}$ as $\hat{M}\underline{N}_0 = \omega\,\underline{N}_0$, together with its adjoint $\hat{M}^{\dagger}\underline{\psi}_0 = \omega\,\underline{\psi}_0$, and we write $\underline{N}$ in the usual factorized form $\underline{N}(t) = P(t)\underline{N}_0$; after projecting on $\underline{\psi}_0$, we obtain the set of equations

$$\begin{cases} \left(\underline{\psi}_0, \underline{N}_0\right) \dfrac{dP}{dt} = \omega \left(\underline{\psi}_0, \underline{N}_0\right) P(t) + \displaystyle\sum_{\alpha=1}^{N_d} \lambda_\alpha \left(\underline{\psi}_0, \underline{\mathbb{C}}^{(\alpha)}\right) \\ \qquad\qquad + \left(\underline{\psi}_0, \underline{S}\right) \\ \dfrac{d}{dt}\left(\underline{\psi}_0, \underline{\mathbb{C}}^{(\alpha)}\right) = \left(\underline{\psi}_0, \hat{d}_\alpha \cdot \underline{N}_0\right) P(t) - \lambda_\alpha \left(\underline{\psi}_0, \underline{\mathbb{C}}^{(\alpha)}\right), \end{cases} \tag{11}$$

where $\hat{d}_\alpha = \beta_\alpha \nu \hat{A}$ and also a source term is introduced, as it is essential for applications to source-driven systems. The case of energy multipoint theory takes a slightly different form, because, as one can realize from equation (10) and previously noted, the concentrations of delayed neutrons in each family are now scalars, the vector character deriving by their probabilities $\underline{\chi}^{(\alpha)}$ to be produced in the given energy group. Then, to be able to project delayed equations on $\underline{\psi}_0$, we are required to multiply each of them by $\underline{\chi}^{(\alpha)}$, which is equivalent to introducing delayed emissivities:

$$\begin{cases} \left(\underline{\psi}_0, \underline{N}_0\right) \dfrac{dP}{dt} = \omega \left(\underline{\psi}_0, \underline{N}_0\right) P(t) + \\ \qquad\qquad + \displaystyle\sum_{\alpha=1}^{N_d} \lambda_\alpha \left(\underline{\psi}_0, \mathbb{C}^{(\alpha)} \underline{\chi}^{(\alpha)}\right) + \left(\underline{\psi}_0, \underline{S}\right) \\ \dfrac{d}{dt}\left(\underline{\psi}_0, \mathbb{C}^{(\alpha)} \underline{\chi}^{(\alpha)}\right) = \left(\underline{\psi}_0, \underline{\chi}^{(\alpha)}\right) \left(\underline{d}^{(\alpha)} \cdot \underline{N}_0\right) P(t) \\ \qquad\qquad - \lambda_k \left(\underline{\psi}_0, \mathbb{C}^{(k)} \underline{\chi}^{(k)}\right); \end{cases} \tag{12}$$

where the matrix $\hat{M}$ is $\left(\hat{M}\right)_{ij} = (\nu - 1)(\beta - 1)\chi_i a_{ij} - C_i \delta_{ij}$ and $\left(\underline{d}^{(\alpha)}\right)_j = \beta_\alpha \nu \sum_i a_{ij}$.

Equations (11) and (12) manifest a point-like structure. Henceforth, the values of the kinetic parameters are included in their coefficients, and can thus be easily retrieved. In an inverse kinetic approach, each flux detector can be simulated with the multipoint equations and the integral parameters may be reconstructed by the projection formulae that are defining the coefficients of Eqs. (11) and (12).

VI. CONCLUSIONS

The paper revisits the basic theory on which the multipoint kinetic method is based. The features of the method are investigated by the analysis of simple multipoint equations based on standard space-discretized and energy multigroup equations. This exercise allows to construct a consistent criticality theory and to gain some physical insight through the observation of the eigenvalue spectrum of the operator.

For realistic applications it is necessary to develop multipoint models in the full space-energy domain: this is particularly important in source-driven problems where neutrons are injected at very high energies in a small volume of the system. This can be naturally done within the proposed approach.

A general physically-based formulation of the multipoint equations is then presented, together with a physical interpretation of the coefficients in terms of probabilities that allows their evaluation, for instance, by direct Monte Carlo simulations. A collapsing procedure, involving a separation-projection procedure, leads to a point-like set of equations. In such a way, by using an inverse approach based on the multipoint model to interpret flux measurements from a multiple detector experiment, the standard integral parameters can be reconstructed. This procedure can be used fruitfully, for instance, in the interpretation of flux measurements for the reactivity prediction and monitoring of accelerator-driven systems, in cases where relevant spatial and spectral effects come into play.

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