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Integral Representations for Various Fission Chain Multiplicity Distributions

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Abstract - *This paper derives integral representations for the multiplicity distribution of neutrons leaked from a multiplying assembly and the multiplicity distribution for those leaked neutrons that are then detected by a measurement system. The probability generating function (PGF) of the leaked neutron distribution is governed by Böhnel's equations, and an equivalent set of equations for the PGF of the detected neutron distribution is also given. This paper presents a method that utilizes functional power series for solving these two sets of equations for the respective PGFs and inverting those PGFs to arrive at the underlying multiplicity distributions.*

I. INTRODUCTION

Neutron multiplication by fission chains is a branching process. In the point-kinetic approximation, all neutrons have the same probability p of inducing fission instead of escaping from the nuclear assembly. In addition, all induced fissions emit independently and identically distributed numbers of fission neutrons. Both the induced fission probability p and the induced fission multiplicity distribution are assumed to be independent of location and incident neutron energy. The basic data for the point-kinetic model is therefore

- p, the probability that any given neutron will induce a fission.
- $q[n]$, the probability that any given induced fission will emit n neutrons.
- $q_s[n]$, the probability that any given spontaneous fission will emit n neutrons.

A random number of neutrons from a single fission chain will manage to avoid inducing any fissions and escape, or "leak", from the nuclear assembly instead. The branching process model can be analyzed to deduce the probability distribution for the number of leaked neutrons. Typically, multiplicity distributions are sought for neutrons leaking from two different types of fission chains:

- $f[n]$, the probability that n neutrons will escape from a fission chain seeded by a single neutron.
- $f_s[n]$, the probability that n neutrons will escape from a fission chain seeded by a random number of neutrons emitted by a single spontaneous fission.

The analysis of branching processes is easier when working with probability generating functions (PGF) rather than directly with probability distributions. The PGF for a general discrete probability distribution $\theta[n]$ is defined to be

$$\Theta(s) = \sum_{n=0}^{\infty} \theta[n] s^n . \tag{1}$$

In all that follows, we will adopt the convention that PGFs are written using capitalized letters, *e.g.* $Q_s(s)$ is the PGF for $q_s[n]$. Along with several other useful properties, the factorial moments for a random variable can be easily computed from a PGF:

$$M_{(r)} = \lim_{z \to 1^-} \frac{d^r}{ds^r} \Theta(s) . \tag{2}$$

In addition, the original probability distribution $\theta[n]$ can be recovered by using [1]

$$\theta[n] = \frac{1}{n!} \frac{d^n}{ds^n} \Theta(s) \bigg|_{s=0} . \tag{3}$$

In the point-kinetic approximation, the PGFs for leaked neutron multiplicities are governed by Böhnel's equations [2]:

$$F(s) = (1-p)s + pQ(F(s)) \tag{4}$$

$$F_s(s) = Q_s(F(s)) . \tag{5}$$

In principle, these functional equations can be solved for $F(s)$ and $F_s(s)$. Then equation (1) can be inverted by any one of several means to yield $f[n]$ and $f_s[n]$.

However, equation (4) is deceptive; it cannot be solved using elementary algebraic manipulations. On the other hand, formula (2) can be applied to equations (4) and (5) to produce a sequence of coupled equations for the distributions' factorial that *can* be solved iteratively by elementary means [2]. For some applications, these factorial moments are sufficient [3]; and this is where the situation remained for many years after the publication of Böhnel's original paper in 1985.

However in recent years, new methods have been applied to Böhnel's equations that yield solutions for the full PGFs. In 2006, Enqvist, *et al*. [4,5] obtained the underlying multiplicity distributions $f[n]$ and $f_s[n]$ from Böhnel's equations in a method similar to Böhnel's method for finding the factorial moments. Enqvist, *et al*. applied formula (3) to Böhnel's equations, rather than formula (2), to get a sequence of coupled equations for $f[n]$ instead of the factorial moments obtained by Böhnel. The first equation in the sequence is a non-linear equation in $f[n]$ alone. For $n \geqslant 1$, each equation is linear in $f[n]$, but non-linear in $f[n']$ for $n' \leqslant n$. When solving each of these equations recursively, all the non-linear terms are already known from the solution of the previous equations. Thus for $n \geqslant 1$, each equation reduces to a linear expression in $f[n]$ alone. The first equation must be solved numerically, but after that, each subsequent equation can be solved algebraically, yielding closed-form expressions for $f[n]$. The $f_s[n]$ multiplicity distribution is obtained in a similar manner. However, the complexities of these expressions mount rapidly. For large n, they can be practically found only by resorting to a symbolic computation software package, such as Mathematica [6].

In 2012, Prasad and Snyderman published a power series solution whose coefficients involve multinomial expansions [7]. The underlying multiplicity distribution is recovered by identifying $f[n]$ with the coefficient for the s^n term in the power series solution for $F(s)$.

Finally, just last year (2016), Chambers, *et al*. published a paper in which they solve Böhnel's equations for $F(s)$ numerically, using a root-finding algorithm [8]. Next, they compute the characteristic function $\phi_f(\theta)$ for $f[n]$ from the PGF by evaluating $\phi_f(\theta) = F\left(e^{i\theta}\right)$. Then the characteristic function for the $f_s[n]$ distribution is computed directly from ϕ_f by $\phi_s(\theta) = Q_s\left(\phi_f(\theta)\right)$. Finally, they recover the multiplicity distributions by applying the fast Fourier transform to $\phi_f(\theta)$ and $\phi_s(\theta)$. Chambers, *et al*. also calculate the measured neutron multiplicity distribution similarly.

Although generating functions lie at the intersection of combinatorics and analysis, Prasad and Snyderman's method is arguably combinatoric at its heart, given its reliance on recursion and multinomial expansions (and also in light of their extensive discussion of its combinatoric properties.). The method of Chambers, *et al*. is essentially algebraic and numerical, whereas the method of Enqvist, *et al*. is mostly algebraic.

This paper presents a fourth method for finding neutron multiplicity distributions that, in contrast, is more purely analytic.

II. SUMMARY OF THE MAIN RESULTS

The application to Böhnel's equations of the method described below (and different from that employed in reference [7]) for finding power series solutions of functional equations leads to the following integral representations for the leaked neutron multiplicity distributions:

$$\begin{aligned} f[0] &= z_o \\ f[n] &= \frac{(1-p)^n}{2\pi n i} \oint_{|w|=1} \frac{dw}{\left(w - pQ(w)\right)^n}, \quad n \geq 1, \end{aligned} \tag{6}$$

and

$$\begin{aligned} f_s[0] &= Q_s(z_o) \\ f_s[n] &= \frac{(1-p)^n}{2\pi n i} \oint_{|w|=1} \frac{Q_s'(w)}{\left(w - pQ(w)\right)^n} dw, \quad n \geq 1, \end{aligned} \tag{7}$$

where z_o is the particular root of the equation

$$z - pQ(z) = 0 \tag{8}$$

that lies in the open disk $|z| < 1$.

These are very well behaved proper integrals; they are evaluated over a finite length contour which never encounters singularities. Therefore they lend themselves to very fast, trouble-free numerical integration. Numerical integration procedures included in standard numerical computation packages are capable of carrying out these quadratures very efficiently.

In addition, the availability of these packages makes the evaluation of equations (6) and (7) extremely easy to implement in code. For instance, the computation of the entire multiplicity distribution, f_s, can be performed in as little as eight lines of MATLAB code [9].

Another attractive feature of these representations is that the dependence on the parameter p is given analytically. Hence the parametric dependence of the neutron multiplicity distributions can be studied analytically. Moreover, derivatives with respect to p can also be taken analytically, which is useful for applications like curve fitting and statistical parameter estimation.

In actual measurements, not every leaked neutron is actually detected, because the detector efficiency ε is less than 100%. Since each neutron detection is the successful outcome of an independent Bernoulli trial, the detected neutron multiplicity distribution is

$$g[m] = \sum_{n=m}^{\infty} \binom{n}{m} \varepsilon^m \left(1-\varepsilon\right)^{n-m} f_s[n]\,. \tag{9}$$

Unfortunately this is an infinite series that converges rather slowly. Therefore an integral representation for $g[m]$, similar to equations (6) and (7), would be useful.

From Böhnel's equations, it is easy to derive functional equations for $G(s)$, the PGF of $g[m]$:

$$F_m(s) = \left(1-p\right)\left(1-\varepsilon+\varepsilon s\right) + pQ\left(F_m(s)\right) \tag{10}$$

$$G(s) = Q_s\left(F_m(s)\right) \tag{11}$$

Then the application of the same methodology leads to the following integral representation for $g[m]$:

$$\begin{aligned} &g[0] = Q_s(z_o) \\ &g[m] = \\ &\quad \frac{\varepsilon^m (1-p)^m}{2\pi m i} \oint_{|w|=1} \frac{Q_s'(w)}{\left[(\varepsilon-1)(1-p)+w-pQ(w)\right]^m} dw \\ &\quad m \geq 1, \end{aligned} \tag{12}$$

where z_o is the particular root of

$$(\varepsilon-1)(1-p)+z-pQ(z)=0 \tag{13}$$

that lies in the open disk $|z|<1$.

III. OUTLINE OF THE DERIVATIONS

1. A Method for Solving Functional Equations

Consider this prototypical problem:

Given two functions, $R_1(s,t)$ and $R_2(t)$, solve the general functional equation

$$R_1\left(s, \Theta(s)\right) = 0 \,, \tag{14}$$

for $\Theta(s)$ and also compute $R_2\left(\Theta(s)\right)$.

Often it is not possible to solve Eq, (14) for Θ(*s*) by elementary means, but sometimes we are fortunate and can rearrange Eq. (14) to arrive at an expression with the form

$$s = H\left(\Theta(s)\right). \tag{15}$$

Clearly, the function *H* is the inverse of the function Θ that we seek. To simplify working with *H* rather than the still unknown Θ, let us make the following change-of-variable

$$z = \Theta(s) \,, \tag{16}$$

so that

$$s = H(z) \,, \tag{17}$$

$$R_2\left(\Theta(s)\right) = R_2(z) \,. \tag{18}$$

Now suppose we could somehow express both the identity function, $u(z) = z$ and $R_2(z)$ in terms of $H(z)$ such that

$$z = u(z) = \mathscr{E}_1\left[H(z)\right], \tag{19}$$

$$R_2(z) = \mathscr{E}_2\left[H(z)\right]. \tag{20}$$

Then we could substitute equations (15) and (16) into equations (19) and (20) to perform a change of variable from *z* back to *s* and thus arrive at expressions for $\Theta(s)$ and $R_2\left(\Theta(s)\right)$:

$$\Theta(s) = \mathscr{E}_1[s], \tag{21}$$

$$R_2\left(\Theta(s)\right) = \mathscr{E}_2[s]. \tag{22}$$

Therefore, all that remains to be done is to find the formal expressions $\mathscr{E}_1$ and $\mathscr{E}_2$. These expressions can take the form of a functional series involving powers of *H*(*z*), which in turn, can be found using the following theorem.

THEOREM 1 *Given*

(a) functions f(z) and H(z),

(b) a closed contour C,

(c) a closed, simply connected domain $\hat{\Omega}$,

let Ω be the interior of C, and

$$\Omega_o = \left\{ z \in \Omega : \left|H(z)\right| < \left|H(w)\right| \;\; \forall w \in C \right\}.$$

If

(i) f(z) and H(z) are analytic everywhere on C and Ω,

(ii) $\hat{\Omega} \subseteq \Omega_o$,

(iii) H(z) has exactly one zero in Ω, and it is simple,

(iv) H(z) has no zeros on C,

(v) $H'(z)$ has no zeros in $\hat{\Omega}$,

then

$$f(z) = f(z_o) + \sum_{n=1}^{\infty} A_n H(z)^n \quad \forall z \in \hat{\Omega} \,, \tag{23}$$

where

$$A_n = \frac{1}{2\pi n i} \oint_C \frac{f'(w)}{H(w)^n} dw \,, \tag{24}$$

and z_o is the zero of H(z) that lies in Ω.

Thus if *H*(*z*) satisfies the criteria set out in this theorem, we can expand $u(z) = z$ and $R_2(z)$ as

$$u(z) = z = z_o + \sum_{n=1}^{\infty} A_n H(z)^n, \tag{25}$$

$$R_2(z) = R_2(z_o) + \sum_{n=1}^{\infty} B_n H(z)^n, \tag{26}$$

where

$$A_n = \frac{1}{2\pi n i}\oint_C \frac{dw}{H(w)^n}, \qquad B_n = \frac{1}{2\pi n i}\oint_C \frac{R_2'(w)}{H(w)^n}\,dw. \tag{27}$$

Then substituting equations (15) and (16) into equations (25) and (26) yields, finally,

$$\left.\begin{aligned}\theta(s) &= z_o + \sum_{n=1}^{\infty} A_n s^n \\ R_2\left(\theta(s)\right) &= R_2(z_o) + \sum_{n=1}^{\infty} B_n s^n\end{aligned}\right\} \quad \forall\, s \in H\left(\hat{\Omega}\right). \tag{28}$$

Theorem 1 is similar to what has come to be known as "Teixeira's theorem" [10,11,12,13], a classical result in the theory of complex analysis. Depending on the author, slightly different preconditions are quoted in the statement of the theorem. Some authors require $f(z)$ and $H(z)$ to be regular, while others only require them to be merely analytic. Some authors state the theorem for a fixed z, while others allow z to range over all of Ω, the interior of the integration contour. Prompted in part by this lack of consistency, Theorem 1 was developed to have preconditions adapted to our particular problem.

2. Finding the Leaked Neutron Multiplicity Distributions

The problem of solving Böhnel's equations (4) and (5) has the same form as the prototypical problem (14). Furthermore, equation (4) can be rearranged to yield

$$s = \frac{F(s) - pQ\left(F(s)\right)}{1-p} \doteq H\left(F(s)\right). \tag{29}$$

Thus we can use the previous section's method. Making the substitution $z = F(s)$ yields the function

$$H(z) = \frac{z - pQ(z)}{1-p}. \tag{30}$$

If we choose any $0 < \delta < 1$ and define

$$\hat{\Omega} = \left\{z : \left|H(z)\right| \le \delta\right\}, \qquad C = \left\{z : |z| = 1\right\}, \tag{31}$$

then it is possible to show that $H(z)$ meets the criteria in Theorem 1. Thus Theorem 1 can be applied, ultimately leading to

$$F(s) = z_o + \sum_{n=1}^{\infty} A_n s^n, \qquad F_s(s) = Q_s \circ F(s) = Q_s(z_o) + \sum_{n=1}^{\infty} B_n s^n, \tag{32}$$

where

$$A_n = \frac{(1-p)^n}{2\pi n i}\oint_{|w|=1} \frac{dw}{\left(w - pQ(w)\right)^n}, \qquad B_n = \frac{(1-p)^n}{2\pi n i}\oint_{|w|=1} \frac{Q_s'(w)}{\left(w - pQ(w)\right)^n}\,dw, \tag{33}$$

and z_o is the root of equation (8).

But by the definition (1) for probability generating functions,

$$F(s) = \sum_{n=0}^{\infty} f[n]s^n, \qquad F_s(s) = \sum_{n=0}^{\infty} f_s[n]s^n. \tag{34}$$

Since power series representations are unique, we can identify the coefficients in equations (34) with the coefficients in equations (32), thus proving equations (6) and (7).

3. Finding the Measured Neutron Multiplicity Distribution

The problem of solving functional equations (10) and (11) also has the same form as the prototypical problem (14). Rearranging equation (10) and making the substitution $z = F_s(s)$ yields

$$s = \frac{z - pQ(z) - (1-p)(1-\varepsilon)}{\varepsilon(1-p)} \doteq \tilde{H}(z). \tag{35}$$

If $C = \{z : |z| = 1\}$ and $\hat{\Omega} = \left\{z : \left|\tilde{H}(z)\right| \le \delta\right\}$ as before, then it can be shown that $\tilde{H}(z)$ meets the criteria in Theorem 1 also. Furthermore, the zero, z_o, of $\tilde{H}(z)$ is the root of equation (13).

To prove equation (12), all the same steps that were followed in solving Böhnel's equations, above, are now repeated here, except that:

(a) $\tilde{H}(z)$ is substituted for $H(z)$.

(b) The root of equation (13) is used for z_o instead of the root of equation (8).

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