M&C 2017 - International Conference on Mathematics & Computational Methods Applied to Nuclear Science & Engineering, Jeju, Korea, April 16-20, 2017, on USB (2017)

Convergence Analysis of Partial Current Based Coarse Mesh Finite Difference Method (p-CMFD) with Two Parameter Relaxation

Chang Je Park[a*], Moon-ghu Park[a], Ser Gi Hong[b]

[a]*Nuclear Engineering Dept., Sejong Univ., 209 Neungdong-ro, Gwangjin-gu, Seoul 05006, South Korea*
[b]*Nuclrear Engineering Dept., Kyung Hee Univ., 1732 Deogyeong-daero, Giheung-gu, Yongin-si, Gyeonggi-do17104, South Korea*
**Corresponding author: parkcj@sejong.ac.kr*

Abstract - *A p-CMFD with two-parameter relaxation is suggested newly to enhance convergence of the p-CMFD and some numerical results are also provided for the slab geometry. Two relaxation parameters are applied when obtaining scalar fluxes of the fine meshes extending from the coarse mesh solution. With the new p-CMFD with two-parameter relaxation, the convergence analysis with Fourier ansatz has been carried out and a few numerical tests are also done to confirm Fourier analysis. Comparison results with CMFD and CMR are also provided.*

I. INTRODUCTION

In order to speed up the convergence of neutron transport equation, lots of acceleration methods have been implemented including diffusion synthetic acceleration (DSA) method and coarse mesh rebalance (CMR) method.[1]-[3] Among them, the partial current based coarse mesh finite difference acceleration method (p-CMFD) has been suggested by extending the conventional coarse mesh finite difference method (CMFD) and the p-CMFD has been applied various application areas of the neutron transport method.[4]-[7] Recently, a relaxation approach of diffusion coefficient was proposed to get stable solutions of coarse mesh finite difference method (CMFD).[8]-[10] It is found out that the relaxation parameter plays important role in stabilization through over-relaxation or extrapolation and in fast convergence through under-relaxation or interpolation.

This paper provides a Fourier analysis for the two-parameter relaxed p-CMFD method, which provides more redundancy compared with the previous one-parameter relaxation method. Some numerical results for a slab-geometry problem are provided including other acceleration methods with two-parameter relaxation.

II. P-CMFD FORMULAR WITH RELAXATION

The high order discrete equation of the slab geometry with diamond differencing method is given as follows.

$$\mu_n \frac{\psi_{k+1/2}^{l+1/2}-\psi_{k-1/2}^{l+1/2}}{h}+\sigma\frac{\psi_{k+1/2}^{l+1/2}+\psi_{k-1/2}^{l+1/2}}{2}=\sigma_s\phi_k^l+Q, \tag{1}$$

$$\phi_k^{l+1/2}=\frac{1}{2}\sum_{n=1}^{N}w_n\frac{\psi_{k+1/2}^{l+1/2}+\psi_{k-1/2}^{l+1/2}}{2}. \tag{2}$$

where l is an iteration index and k is an fine mesh index and others are typical notations are used.[4] The low order partial current based coarse mesh finite difference equation is given as

$$J_{i+1/2}^{l+1/2}=\frac{1}{2}\sum_{n=1}^{N}w_n\ \mu_n\ \psi_{n,i+1/2}^{l+1/2}. \tag{3}$$

$$J_{i+1/2}^{+,l+1/2}=\frac{1}{2}\sum_{n=1}^{N/2}w_n|\mu_n|\psi_{n,i+1/2}^{l+1/2}. \tag{4}$$

$$J_{i+1/2}^{-,l+1/2}=\frac{1}{2}\sum_{n=N/2+1}^{N}w_n|\mu_n|\psi_{n,i+1/2}^{l+1/2}. \tag{5}$$

$$J_{i+1/2}^{+,l+1}=\frac{-\tilde{D}_{i+1/2}\left(\phi_{i+1}^{l+1/2}-\phi_i^{l+1/2}\right)+2\hat{D}_{i+1/2}^{l+1,+}\phi_i^{l+1/2}}{2} \tag{6}$$

$$J_{i+1/2}^{-,l+1}=\frac{\tilde{D}_{i+1/2}\left(\phi_{i+1}^{l+1/2}-\phi_i^{l+1/2}\right)+2\hat{D}_{i+1/2}^{l+1,-}\phi_{i+1}^{l+1/2}}{2} \tag{7}$$

$$\hat{D}_{i+1/2}^{l+1,+}=\frac{2J_{i+1/2}^{+,l+1/2}+\tilde{D}_{i+1/2}(\phi_{i+1}^{l+1/2}-\phi_i^{l+1/2})}{2\phi_i^{l+1/2}}, \tag{8}$$

$$\hat{D}_{i+1/2}^{l+1,-}=\frac{2J_{i+1/2}^{-,l+1/2}-\tilde{D}_{i+1/2}(\phi_{i+1}^{l+1/2}-\phi_i^{l+1/2})}{2\phi_{i+1}^{l+1/2}}, \tag{9}$$

where i denotes the coarse mesh index.

In the CMFD, the net current is adjusted by one correction factor. The final form of p-CMFD equation with relaxation is

$$\begin{aligned}&-\tilde{D}_{i+1/2}\left(\phi_{i+1}^{l+1}-\phi_i^{l+1}\right)-(\hat{D}_{i+1/2}^{l+1,-}\phi_{i+1}^{l+1}-\hat{D}_{i+1/2}^{l+1,+}\phi_i^{l+1})\\&+\tilde{D}_{i-1/2}\left(\phi_i^{l+1}-\phi_{i-1}^{l+1}\right)+(\hat{D}_{i-1/2}^{l+1,-}\phi_i^{l+1}-\hat{D}_{i-1/2}^{l+1,+}\phi_{i-1}^{l+1})\\&+h_i(\sigma_i-\sigma_{si})\phi_i^{l+1}=h_i\,q_i.\end{aligned} \tag{10}$$

The two relaxation parameters(α, β) are applied to obtain fine mesh fluxes from the coarse mesh fluxes which are solution of Eq. (10).

$$\phi_k^{l+1}=\alpha\,\phi_k^{l+1/2}\frac{\phi_i^{l+1}}{\frac{1}{p}\sum_k\phi_k^{l+1/2}}+(1-\alpha)\phi_k^l+\beta(\phi_k^l-\phi_k^{l-1}), \tag{11}$$

$$\phi_i^{l+1/2}=\frac{1}{p}\sum_k\phi_k^{l+1/2}, \tag{12}$$

where p is the number of fine meshes in a coarse mesh as shown in Fig. 1

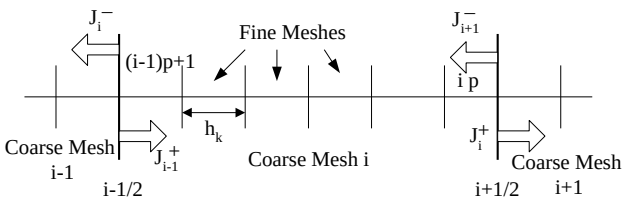

Fig. 1. Index of coarse and fine meshes.

The first relaxation parameter (α) is widely used to obtain fast extrapolated solution. The second relaxation parameter (β) is suggested newly to enhance the convergence of p-CMFD by extending the differences of the previous two scalar fluxes. This idea is similar as the two-parameter source extrapolation in acceleration of k eigenvalue in the power method.[9] When the first and the second parameters are unit and zero, respectively, no relaxation is applied to the p-CMFD. In the case of the other coarse mesh rebalance methods such as CMR and CMFD, two parameter relaxation of Eq.(11) is applied easily without additional modifications.

III. CONVERGENCE ANALYSIS OF P-CMFD FORMULAR WITH RELAXATION

The Fourier ansatz are applied to linearize the p-CMFD equation with two-parameter relaxation as follows [4]

$$\phi_i^{l+1} = Q/\sigma_a(1+\varepsilon\varsigma_i^{l+1}), \tag{13}$$

$$\phi_i^{l+1/2} = Q/\sigma_a(1+\varepsilon\varsigma_i^{l+1/2}), \tag{14}$$

$$\psi_{n,i+1/2}^{l+1/2} = Q/\sigma_a(1+\varepsilon\xi_{n,i+1/2}^{l+1/2}). \tag{15}$$

Then the linearized p-CMFD with relaxation is obtained by taking the first order of $O(\varepsilon)$ with the simple arithmetic derivation.

$$\mu_n\frac{\xi_{n,k+1/2}^{l+1/2}-\xi_{n,k-1/2}^{l+1/2}}{h}+\frac{\xi_{n,k+1/2}^{l+1/2}+\xi_{n,k-1/2}^{l+1/2}}{2}=c\varsigma_k^l, \tag{16}$$

$$\varsigma_k^{l+1/2}=\frac{1}{2}\sum_{n=1}^{N}w_n\frac{\xi_{n,k+1/2}^{l+1/2}+\xi_{n,k-1/2}^{l+1/2}}{2}, \tag{17}$$

$$\begin{aligned}&(\frac{2}{3ph}+\gamma)\varsigma_i^{l+1}-(\frac{1}{3ph}+\gamma/2)\varsigma_{i+1}^{l+1}-(\frac{1}{3ph}+\gamma/2)\varsigma_{i-1}^{l+1}+h\sigma_a\varsigma_i^{l+1}\\&=(\frac{2}{3ph}+\gamma)\varsigma_i^{l+1/2}-(\frac{1}{3ph}+\gamma/2)\varsigma_{i+1}^{l+1/2}-(\frac{1}{3ph}+\gamma/2)\varsigma_{i-1}^{l+1/2}\\&\quad-h\sigma\sum_k\varsigma_k^{l+1/2}+h\sigma_s\sum_k\varsigma_k^l,\end{aligned} \tag{18}$$

$$\varsigma_k^{l+1}+\varsigma_i^{l+1/2}=\alpha(\varsigma_k^{l+1/2}+\varsigma_i^{l+1})+(1-\alpha)(\varsigma_k^l+\varsigma_i^{l+1/2})+\beta(\varsigma_k^l-\varsigma_k^{l-1}). \tag{19}$$

where

$$\gamma=\frac{1}{2}\sum_{n=1}^{N/2}w_n|\mu_n| \tag{20}$$

To obtain spectral radius of iterative scheme, Fourier ansatz are defined as follows

$$\varsigma_i^l=\omega^l C\exp(j\lambda x_i), \tag{21}$$

$$\varsigma_k^l=\omega^l A_k\exp(j\lambda x_k),\quad A_k=A_{k+p} \tag{22}$$

$$\varsigma_i^{l+1/2}=\omega^l B\exp(j\lambda x_i), \tag{23}$$

$$\varsigma_k^{l+1/2}=\omega^l B_k\exp(j\lambda x_k),\quad B_k=B_{k+p} \tag{24}$$

$$\xi_{n,k+1/2}^{l+1/2}=\omega^l a_{n,k}\exp(j\lambda x_{k+1/2}),\quad a_{n,k}=a_{n,k+p} \tag{25}$$

From Eqs.(16) and (17), the following relation is derived definitely by the following the coarse mesh rebalance case.[2]

$$\mathbf{B}=\mathbf{H}\,\mathbf{A}, \tag{26}$$

where

$\mathbf{B}=[B_{(i-1)p+1}\cdots B_{ip}]^T$, $\mathbf{A}=[A_{(i-1)p+1}\cdots A_{ip}]^T$,

$$\mathbf{M}=\begin{pmatrix}0&1&&&0\\&0&1&&\\&\cdot&\cdot&\cdot&\\&&&&1\\1&&&&0\end{pmatrix},\ \mathbf{I}=\begin{pmatrix}1&0&&&0\\&1&0&&\\&\cdot&\cdot&\cdot&\\&&&1&0\\0&&&&1\end{pmatrix},$$

$$\mathbf{K}=\frac{1}{j}[\exp(j\tau)\mathbf{M}-\exp(-j\tau)\mathbf{I}]\times[\exp(j\tau)\mathbf{M}+\exp(-j\tau)\mathbf{I}]^{-1},$$

$$\mathbf{H}=\frac{1}{2}\sum_{n=1}^{N}w_n[\mathbf{I}+(2\mu_n/h)^2\mathbf{K}\cdot\mathbf{K}]^{-1}.$$

From Eq. (18), we can obtain the matrix equation as[4]

$$\omega\mathbf{A}=\mathbf{B}+\theta\mathbf{VUB}-\theta\mathbf{VUC}=\mathbf{LC} \tag{27}$$

where

$\mathbf{V}=[\exp(j\tau(1-p))\cdots\exp(j\tau(p-1))]$, $\mathbf{U}=[\exp(j\tau(p-1))\cdots\exp(j\tau(1-p))]^T$,

$$\mathbf{L}=\mathbf{H}+\theta\mathbf{VUH}-\theta\mathbf{VUI},$$

$$\theta=hc/[\{2\sin^2(p\tau)\}\eta+hp(1-c)].$$

$$\eta=\frac{hc}{2\gamma\sin^2(p\tau)+ph(1-c)},\quad \tau=\frac{\lambda h}{2},\quad c=\frac{\sigma_s}{\sigma},\quad \gamma=\sum_{n=1}^{N/2}\mu_n w_n. \tag{28}$$

Eq.(19) is changed into the matrix form after simple steps

$$\omega^2\mathbf{C}=\omega\mathbf{A}(\omega\alpha+1-\alpha)+\beta(\omega-1)\mathbf{C} \tag{29}$$

Combining Eqs.(27) and (29), the final form of eigenvalue problem is obtained, which is similar formula as those of results the original p-CMFD.[4]

$$\bar{\omega}\mathbf{C}=\mathbf{LC} \tag{30}$$

where

$$\bar{\omega}=\frac{\omega^2-\beta\omega+\beta}{\omega\alpha+1-\alpha}. \tag{31}$$

The eigenvalue is obtained by solving the quadratic polynomial equation as

$$\omega^2+b\omega+c=0 \tag{32}$$

where

$$b=-\beta-\alpha\bar{\omega},\quad c=\beta-\bar{\omega}+\alpha\bar{\omega}. \tag{33}$$

The spectral radius is given as

$$\rho=\sup_\tau|\omega(\tau)|. \tag{34}$$

In the case of CMFD and CMR with two parameter relaxation, the same results are derived but the parameter in Eq. (28)[4], that is,

CMR : $\eta=\gamma$,

CMFD : $\eta=2/(3\sigma ph)$,

p-CMFD : $\eta=2/(3\sigma ph)+\gamma$.

Table II shows the trend of spectral radius by changing two parameters. From the results, it is found that for a fixed α , the optimal β is existing, which provides smaller spectra(ω) radius than p-CMFD without relaxation($\bar{\omega}$). And it is also found that the spectral radius may decrease by choosing combination of the larger α and the smaller β parameters. The optimal selection of two parameters is under investigation.

Table I. Spectral Radius for p-CMFD With and Without Relaxation

α	β	$\bar{\omega}$ [a]	ω [b]
0.5	0.0	0.7	0.79
0.5	0.0	0.9	0.93
0.5	0.6	0.9	0.88
0.5	1.0	0.9	0.74
0.5	1.2	0.9	0.87
0.8	0.0	0.7	0.75
0.8	0.0	0.9	0.92
0.8	0.6	0.9	0.78
0.8	1.0	0.9	0.90
0.8	1.2	0.9	1.00
1.0	0.0	0.9	0.90
1.0	0.7	0.9	0.84
1.0	1.0	0.9	1.00
1.2	0.3	0.9	0.81
1.2	0.7	0.9	0.93

[a]: p-CMFD without relaxation,
[b]: p-CMFD with relaxation.

IV. RESULTS

The test problem is chosen as the one dimensional slab problem as shown in Fig. 2.[4]

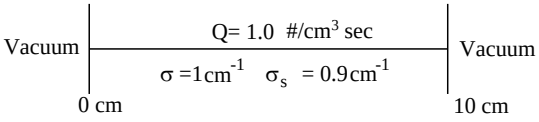

Fig. 2. Configuration of 1D slab test problem.

The convergence criterion is 1.0E-9 for the maximum fine mesh scalar flux. For spatial discretization, the diamond differencing scheme is used and S_{16} Gauss-Legendre quadrature set is used for angular discretization.

Table II shows the solution of the source iteration(SI), CMR, CMFD, p-CMFD with and without relaxation. Various combinations of two parameters are tested in order to provide fast convergence based on analysis results. Without relaxation, the number of iteration decreases as the optical size (σh) increases and the trends of three different coarse mesh rebalance methods are slightly different due to their convergence characteristics, which is consistent to the previous analysis.[4] However the proper relaxation provides quite different results. When the number of fine meshes is same as that of coarse meshes (p=1), all three methods provided convergent solution with two parameter relaxation. Especially, CMR provides the best results when number of coarse mesh is 5. The iteration number of the CMFD acceleration is very large, which comes from unadjusted relaxation parameters for CMFD only. In this test, two parameters are adjusted roughly for the p-CMFD. Thus, p-CMFD provides better robust and stabilized convergence. When p increases up to 4, the divergent region of CMFD becomes convergent due to relaxation. The CMR results are enhanced up to 4 coarse meshes but for 2 coarse mesh problem, it is not converged either. The performance of p-CMFD results are enhanced with two-parameter relaxation and it may well provide better convergence when the relaxation parameters are optimized.

Table I. Number of Iterations and Numerical Spectral Radius (c=0.9)

p	Relaxation	σh [a] (I [b])	SI	CMR	CMFD	p-CMFD
1	No Relax-ation	0.1 (100)	145 [c] 0.8781 [d]	23 0.4248	12 0.1758	12 0.1908
	α=1 β=0	1.0 (10)	144 0.8779	22 0.4041	22 0.4041	13 0.1905
		2.0 (5)	143 0.8772	N.C. [e]	N.C.	32 0.5269
	α=0.8 β=0.04	1.0 (10)	-	128 0.4000	14 0.1848	15 0.3169
		2.0 (5)	-	7 0.0400	113 0.8384	14 0.1217
4	No Relax-ation	0.625 (4)	-	22 0.4041	48 0.6613	66 0.7538
	α=1 β=0	1.25 (2)	-	N.C.	N.C.	62 0.7394
	α=0.8 β=0.3	0.625 (4)	-	18 0.3000	34 0.2397	34 0.3078
		1.25 (2)	-	N.C.	36 0.6387	36 0.7165

[a]:Total cross section (σ) x Mesh size of fine mesh (h), [b]:Number of coarse meshes, [c]:Number of iterations, [d]: Numerical spectral radius, [e]: Not converged.

IV. CONCLUSIONS

This paper provides convergence analysis for p-CMFD with two-parameter relaxation. The relaxation is carried out when obtaining scalar fluxes of fine meshes after coarse mesh rebalancing. Two parameters are taken into consideration including the two previous step scalar fluxes. From the convergence analysis, the eigenvalue matrix formula becomes in the same framework of the original p-CMFD. Therefore, it is easy to represent the general forms of coarse mesh rebalance methods such as CMR and CMFD methods in the same framework. From the simple numerical tests of a slab-geometry problem, it is confirmed that the relaxation becomes a way to extend convergence region of various coarse mesh rebalance methods.

As a future work, the optimal selection of relaxation parameter will be studied. One of solution is to apply the advanced optimization approach. If possible to express

arithmetic equation, non-linear numerical search algorithms are good candidates.

NOMENCLATURE

μ = polar angle cosine, cosθ
ψ = angular flux,
ϕ = scalar flux,
J = net current,
J^+ = positive partial current,
J^- = negative partial current,
σ = macroscopic total cross section,
σ_s= macroscopic scattering cross section,
σ_a= macroscopic absorption cross section,
Q= external source.

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