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DFEM Discretization of Quasidiffusion Moment Equations in 1D Slab Geometry

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Abstract - *In this paper, we develop a linear discontinuous finite element method (DFEM) for the spatial approximation of the low-order quasidiffusion (QD) equations in 1D slab geometry. This discretization is consistent with a linear DFEM discretization of the transport equation. It involves special interface conditions at cell edges based the idea of QD boundary conditions. The proposed method is studied by means of test problems formulated with the method of manufactured solutions. We present numerical results demonstrating the performance of the proposed method.*

I. INTRODUCTION

The quasidiffusion (QD) method is an efficient method for solving the particle transport equation [1, 2]. It is formulated as a system of high-order transport equation and low-order QD (LOQD) equations. If the LOQD and high-order equations are approximated with an algebraically consistent discretization, then the QD method is a pure acceleration scheme, but the LOQD and high-order equations are not required to be discretized consistently. The method is stable and rapidly converges in the case of independent discretization schemes [3]. Application of such a discretization can improve the accuracy of the transport solution [3, 4], for example, if a high-order accurate scheme for spatial discretization of the LOQD equations is used. From another viewpoint, if within the framework of the QD method, one applies a transport discretization of high-order accuracy, then the discretization of the LOQD equations should at least match the order of the transport scheme to preserve the level of quality of the high-order transport solution.

In this paper we consider the transport scheme defined by the linear discontinuous finite element method (DFEM) to which we apply the QD method. The DFEM transport spatial approximation has 3$^{\text{rd}}$ order accuracy. Hence, to preserve the high quality of the DFEM transport solution, the spatial discretization of the LOQD equations should match this order of accuracy. To achieve this, we develop a linear DFEM for the LOQD equations. To formulate this method we follow the way that is applied in derivation of independent discretization schemes. The DFEM for the LOQD equations thus obtained can lead to a scheme consistent with the DFEM transport discretization, depending on the closure relations and the definition of the nonlinear factors. We formulate two different closures, one of which gives rise to a fully-consistent DFEM.

The slab geometry transport problem with isotropic scattering is given by

$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\sigma_t(x)\psi(x,\mu) = \frac{1}{2}\sigma_s(x)\int_{-1}^{1}\psi(x,\mu')+q(x,\mu)\,, \quad (1)$$

$$\psi(0,\mu) = \psi_{in}^{+}(\mu) \quad \text{for } \mu > 0\,, \quad (2)$$

$$\psi(X,\mu) = \psi_{in}^{-}(\mu) \quad \text{for } \mu < 0\,. \quad (3)$$

The QD method for 1D slab geometry is defined by the high-order transport equation [1]

$$\mu\frac{\partial\psi(x,\mu)}{\partial x} + \sigma_t\psi(x,\mu) = \frac{1}{2}\sigma_s\phi(x) + q(\mu)\,, \quad (4)$$

and the low-order QD (LOQD) equations for the moments of the angular flux

$$\frac{dJ(x)}{dx} + \sigma_a\phi(x) = q_0\,, \quad (5a)$$

$$\frac{d}{dx}(E(x)\phi(x)) + \sigma_t J(x) = q_1\,, \quad (5b)$$

where

$$\phi = \int_{-1}^{1}\psi d\mu\,, \quad J = \int_{-1}^{1}\mu\psi d\mu\,, \quad (6)$$

$$q_n = \int_{-1}^{1}\mu^n q(\mu)d\mu\,. \quad (7)$$

The system of the high-order and low-order equations is closed by the QD, or Eddington, factor

$$E = \frac{\int_{-1}^{1}\mu^2\psi d\mu}{\int_{-1}^{1}\psi d\mu}\,. \quad (8)$$

The boundary conditions for the LOQD equations (5) are defined as follows [5]:

$$J(0) = C_0^{-}(\phi(0) - \phi_{in}^{+}) + J_{in}^{+}\,, \quad (9a)$$

$$J(X) = C_X^{+}(\phi(X) - \phi_{in}^{-}) + J_{in}^{-}\,, \quad (9b)$$

where the boundary factors are

$$C_0^{-} = \left.\frac{\int_{-1}^{0}\mu\psi d\mu}{\int_{-1}^{0}\psi d\mu}\right|_{x=0}\,, \quad C_X^{+} = \left.\frac{\int_{0}^{1}\mu\psi d\mu}{\int_{0}^{1}\psi d\mu}\right|_{x=X}\,, \quad (10)$$

and

$$\phi_{in}^{\pm} = \pm\int_{0}^{\pm 1}\psi_{in}^{\pm}d\mu\,, \quad J_{in}^{\pm} = \pm\int_{0}^{\pm 1}\mu\psi_{in}^{\pm}d\mu\,. \quad (11)$$

To formulate the DFEM discretization of the LOQD equations we start from the differential moment equations in the unclosed form for the zeroth, first and second moments of the angular flux [6]. Subsequently, QD closures will be defined for the unclosed moment equations in a way that is consistent with the DFEM transport approximation. That is, we formulate auxiliary conditions for the scalar flux and current at cell interfaces assuming the discrete high-order transport solution is discontinuous at cell edges. Note that the LOQD equations (5) reduce to the P_1 equations if the angular flux is a linear function of μ. We derive the discretization scheme for the LOQD equations that results in the linear DFEM for the diffusion equation in that case [7]. This scheme can also be used with different transport methods that will lead to independent discretization schemes [3]. To demonstrate the performance of our QD discretization, we present numerical results of several analytic tests formulated with the method of manufactured solutions (MMS).

The remainder of this paper is organized as follows. In Sec. II, we derive discretization of the LOQD equations, formulate cell-interface conditions, and define closure relations and discontinuous QD factors. In Sec. III, we present numerical results. We conclude with a summary in Sec. IV.

II. APPROXIMATION OF EQUATIONS

1. Discretization of the LOQD Equations

To derive the spatial discretization of the LOQD equations, we consider the system of unclosed moment equations

$$\frac{dJ(x)}{dx} + \sigma_a \phi(x) = q_0 \,, \tag{12}$$

$$\frac{dF(x)}{dx} + \sigma_t J(x) = q_1 \,, \tag{13}$$

where

$$F = \int_{-1}^{1} \mu^2 \psi d\mu \,. \tag{14}$$

The DFEM expansion of the low-order solution in the i-th spatial cell ($x_{i-1} \le x \le x_i$) is given by

$$\phi_i(x) = \sum_{c=L,R} \phi_{c,i} B_{c,i}(x) \,, \quad J_i(x) = \sum_{c=L,R} J_{c,i} B_{c,i}(x) \,, \tag{15}$$

where the basis functions are

$$B_{L,i} = \frac{1}{\Delta x_i}(x_i - x) \,, \quad B_{R,i} = \frac{1}{\Delta x_i}(x - x_{i-1}) \,, \tag{16}$$

$$\Delta x_i = x_i - x_{i-1} \,. \tag{17}$$

We integrate Eqs. (12) and (13) with $B_{c,i}$ to get

$$-J^b(x_{i-1}) + \frac{1}{2}(J_{L,i} + J_{R,i}) + \sigma_{a,i}\frac{\Delta x_i}{6}\Big(2\phi_{L,i} + \phi_{R,i}\Big) = \frac{\Delta x_i}{6}\Big(2q_{0,L,i} + q_{0,R,i}\Big) , \tag{18a}$$

$$J^b(x_i) - \frac{1}{2}(J_{L,i} + J_{R,i}) + \sigma_{a,i}\frac{\Delta x_i}{6}\Big(\phi_{L,i} + 2\phi_{R,i}\Big) = \frac{\Delta x_i}{6}\Big(q_{0,L,i} + 2q_{0,R,i}\Big) , \tag{18b}$$

$$-F^b(x_{i-1}) + \frac{1}{\Delta x_i}\int_{x_{i-1}}^{x_i} F_i(x)dx + \sigma_{t,i}\frac{\Delta x_i}{6}\Big(2J_{L,i} + J_{R,i}\Big) = \frac{\Delta x_i}{6}\left(2q_{1,L,i} + q_{1,R,i}\right) , \tag{19a}$$

$$F^b(x_i) - \frac{1}{\Delta x_i}\int_{x_{i-1}}^{x_i} F_i(x)dx + \sigma_{t,i}\frac{\Delta x_i}{6}\Big(J_{L,i} + 2J_{R,i}\Big) = \frac{\Delta x_i}{6}\left(q_{1,L,i} + 2q_{1,R,i}\right) , \tag{19b}$$

where the superscript b indicates terms evaluated at cell interfaces. The closures for Eqs. (18) and (19) are formulated for the terms with the second moment F. The integral term in Eqs. (19) is cast as

$$\int_{x_{i-1}}^{x_i} F_i(x)dx = \bar{E}_i \int_{x_{i-1}}^{x_i} \phi_i(x)dx \,, \tag{20}$$

where the cell-average QD (Eddington) factor is given by

$$\bar{E}_i = \frac{\int_{x_{i-1}}^{x_i} dx \int_{-1}^{1} d\mu \mu^2 \psi_i(x,\mu)}{\int_{x_{i-1}}^{x_i} dx \int_{-1}^{1} d\mu \psi_i(x,\mu)} \,. \tag{21}$$

Here $\psi_i(x,\mu)$ is the angular flux in the i-th cell defined by the solution of the DFEM discretization of the high-order transport equation.

To match the structure of the DFEM diffusion discretization, the cell-interface terms in Eqs. (19) should be closed by the scalar flux, and the QD factor is defined by the corresponding upwinding angular fluxes [7]. The closure is defined as follows:

$$F^b(x_i) = E_i^b \phi^b(x_i) \,, \tag{22}$$

where

$$E_i^b = \left. \frac{\int_{-1}^{0} \mu^2 \psi_{i+1} d\mu + \int_0^1 \mu^2 \psi_i d\mu}{\int_{-1}^{0} \psi_{i+1} d\mu + \int_0^1 \psi_i d\mu} \right|_{x=x_i} \,. \tag{23}$$

Thus, the moment equations (19) with the closures (20) and (22) lead to the discretized first-moment QD equations of the form

$$-E_{i-1}^b \phi^b(x_{i-1}) + \frac{1}{2}\bar{E}_i(\phi_{L,i} + \phi_{R,i}) + \sigma_{t,i}\frac{\Delta x_i}{6}\Big(2J_{L,i} + J_{R,i}\Big) = \frac{\Delta x_i}{6}\left(2q_{1,L,i} + q_{1,R,i}\right) , \tag{24a}$$

$$E_i^b \phi^b(x_i) - \frac{1}{2}\bar{E}_i(\phi_{L,i} + \phi_{R,i}) + \sigma_{t,i}\frac{\Delta x_i}{6}\Big(J_{L,i} + 2J_{R,i}\Big) = \frac{\Delta x_i}{6}\left(q_{1,L,i} + 2q_{1,R,i}\right) . \tag{24b}$$

As a result, the LOQD equations discretized with DFEM are given by Eqs. (18) and (24).

2. Cell-Interface Conditions

The DFEM low-order solution at the cell interface x_i is defined with partial currents from adjacent cells. Hence, the current and scalar flux are given by

$$J^b(x_i) = J_i^+(x_i) + J_{i+1}^-(x_i)\,, \tag{25}$$

$$\phi^b(x_i) = \frac{J_i^+(x_i)}{C_i^+(x_i)} + \frac{J_{i+1}^-(x_i)}{C_{i+1}^-(x_i)}\,, \tag{26}$$

where

$$J_i^\pm(x) = \pm\int_0^{\pm1} \mu\psi_i(x,\mu)d\mu \tag{27}$$

are the partial currents and

$$C_i^-(x) = \frac{\int_{-1}^0 \mu\psi_i(x,\mu)d\mu}{\int_{-1}^0 \psi_i(x,\mu)d\mu}\,, \tag{28a}$$

$$C_i^+(x) = \frac{\int_0^1 \mu\psi_i(x,\mu)d\mu}{\int_0^1 \psi_i(x,\mu)d\mu} \tag{28b}$$

are the interface QD factors defined by the high-order transport solution. To derive conditions at the cell interfaces, we apply the idea of the QD boundary conditions (9) [5]. First, we formulate the cell-interface condition at x_i defined by the solution in the i^{th} cell to obtain

$$J_i(x_i) = C_i^-(x_i)\left(\phi_i(x_i) - \frac{J_i^+(x_i)}{C_i^+(x_i)}\right) + J_i^+(x_i)\,. \tag{29}$$

This condition is used to define the partial current from the i^{th} cell to the $(i+1)^{\text{th}}$ cell. We now introduce a condition at the left boundary of the $(i+1)^{\text{th}}$ cell that provides the partial current from the $(i+1)^{\text{th}}$ cell to the i^{th} cell. This interface relation at x_i is defined by the solution in the $(i+1)^{\text{th}}$ cell and has the form

$$J_{i+1}(x_i) = C_{i+1}^+(x_i)\left(\phi_{i+1}(x_i) - \frac{J_{i+1}^-(x_i)}{C_{i+1}^-(x_i)}\right) + J_{i+1}^-(x_i)\,. \tag{30}$$

We use the interface condition (29) to get the outgoing partial current from the i^{th} cell

$$J_i^+(x_i) = \gamma_i^+(x_i)\Big(J_i(x_i) - C_i^-(x_i)\phi_i(x_i)\Big)\,, \tag{31}$$

where

$$\gamma_i^+(x) = \frac{C_i^+(x)}{C_i^+(x) - C_i^-(x)}\,. \tag{32}$$

The boundary condition (30) leads to the outgoing partial current from the $(i+1)^{\text{th}}$ cell

$$J_{i+1}^-(x_i) = \gamma_{i+1}^-(x_i)\Big(J_{i+1}(x_i) - C_{i+1}^+(x_i)\phi_{i+1}(x_i)\Big)\,, \tag{33}$$

where

$$\gamma_{i+1}^-(x) = \frac{C_{i+1}^-(x)}{C_{i+1}^-(x) - C_{i+1}^+(x)}\,. \tag{34}$$

Substituting Eqs. (31) and (33) into Eqs. (25) and (26) we obtain the cell-interface conditions

$$J^b(x_i) = \gamma_i^+(x_i)\Big(J_i(x_i) - C_i^-(x_i)\phi_i(x_i)\Big) + \gamma_{i+1}^-(x_i)\Big(J_{i+1}(x_i) - C_{i+1}^+(x_i)\phi_{i+1}(x_i)\Big)\,, \tag{35}$$

$$\phi^b(x_i) = \frac{1}{C_i^+(x_i) - C_i^-(x_i)}\Big(J_i(x_i) - C_i^-(x_i)\phi_i(x_i)\Big) + \frac{1}{C_{i+1}^-(x_i) - C_{i+1}^+(x_i)}\Big(J_{i+1}(x_i) - C_{i+1}^+(x_i)\phi_{i+1}(x_i)\Big)\,. \tag{36}$$

We note that if $\psi > 0$ then $C_i^- \leq 0$ and $C_i^+ \geq 0$. The case $C_i^- = C_i^+ = 0$ takes place if $\psi = \delta(\mu)$. Thus $\gamma_i^\pm$ are well defined almost everywhere.

3. Definition of QD Factors Using DFEM Transport Scheme

We now consider the DFEM transport scheme for discretizing the high-order transport equation. The DFEM angular flux spatial expansion is given by

$$\psi_{m,i}(x) = \sum_{c=L,R} \psi_{c,m,i}B_{c,i}(x)\,. \tag{37}$$

We substutute Eq. (37) into Eqs. (21) and (23) to get the QD factors for DFEM in the i^{th} cell

$$\bar{E}_i = \frac{\sum_m \mu_m^2\Big(\psi_{L,m,i} + \psi_{R,m,i}\Big)w_m}{\sum_m\Big(\psi_{L,m,i} + \psi_{R,m,i}\Big)w_m}\,, \tag{38}$$

$$E^b(x_i) = \frac{\sum_{m^-}\mu_m^2\psi_{L,m,i+1}w_m + \sum_{m^+}\mu_m^2\psi_{R,m,i}w_m}{\sum_{m^-}\psi_{L,m,i+1}w_m + \sum_{m^+}\psi_{R,m,i}w_m}\,, \tag{39}$$

where $m^\pm = \{m : \mu_m \gtrless 0\}$. The interface QD factors are given by

$$C_i^\pm(x_i) = \frac{\sum_{m^\pm}\mu_m\psi_{R,m,i}w_m}{\sum_{m^\pm}\psi_{R,m,i}w_m}\,, \tag{40}$$

$$C_{i+1}^\pm(x_i) = \frac{\sum_{m^\pm}\mu_m\psi_{L,m,i+1}w_m}{\sum_{m^\pm}\psi_{L,m,i+1}w_m}\,. \tag{41}$$

It can be shown that the definition of the QD grid functionals (39) and (38) lead to a discretization of the LOQD equations that is consistent with the DFEM discretization of the transport equation.

4. Alternate Discretization

We now formulate an alternate DFEM method, which is not fully consistent. In this case, we use linear approximation of the second moment in the i^{th} cell of the following form:

$$F_i(x) = E_{L,i}\phi_{L,i}B_L(x) + E_{R,i}\phi_{R,i}B_R(x). \tag{42}$$

The first moment equations on the left and right are defined as follows:

$$-E_{L,i}\phi^b(x_{i-1}) + \frac{1}{2}\left(E_{L,i}\phi_{L,i} + E_{R,i}\phi_{R,i}\right) + \sigma_{t,i}\frac{\Delta x_i}{6}\left(2J_{L,i} + J_{R,i}\right) = \frac{\Delta x_i}{6}\left(2q_{1,L,i} + q_{1,R,i}\right)\,, \tag{43a}$$

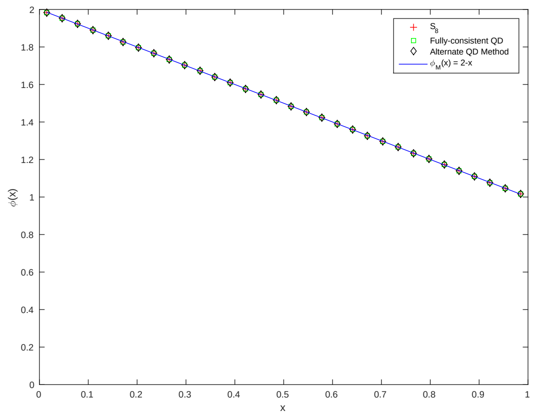

Fig. 1: Linear Solutions.

$$E_{R,i}\phi^b(x_i) - \frac{1}{2}\left(E_{L,i}\phi_{L,i} + E_{R,i}\phi_{R,i}\right) + \sigma_{t,i}\frac{\Delta x_i}{6}\left(J_{L,i} + 2J_{R,i}\right) = \frac{\Delta x_i}{6}\left(q_{1,L,i} + 2q_{1,R,i}\right). \quad (43b)$$

The QD factors are defined as

$$E_{L,i} = \frac{\sum_m w_m \mu_m^2 \psi_{L,i}}{\sum_m w_m \psi_{L,i}}, \quad (44)$$

$$E_{R,i} = \frac{\sum_m w_m \mu_m^2 \psi_{L,i}}{\sum_m w_m \psi_{L,i}}. \quad (45)$$

The LOQD equations discretized with the alternate DFEM method are given by Eqs. (18) and (43).

III. NUMERICAL EXPERIMENTS

1. Linear MMS solution

The manufactured solution is linear in space, and we would expect that the linear DFEM discretization will obtain a linear solution exactly. The manufactured angular flux solution for this problem is taken to be

$$\psi_M(x,\mu) = \frac{3}{64}(2-x)\left(9+5\mu^2\right),$$

for $x \in [0,1]$. With this angular flux, the manufactured scalar flux and QD factor are

$$\phi_M(x) = 2 - x, \quad E_M(x) = \frac{3}{8}.$$

This MMS problem is solved with $\sigma_t = 1$, $\sigma_s = 0.9$ and S_8 Gauss-Legendre quadrature. Figure 1 shows that the S_8 DFEM transport solution, from which the factors for the QD methods are computed, and the QD solution all lie on the linear solution $\phi_M(x)$. Thus, we see that solutions with the fully-consistent and alternate methods both retain the linear solution as expected. A calculation of the discrete L_2-norm of the difference between the numerical and manufactured solutions was observed to be on the order of the machine precision.

2. Analytic Tests

To evaluate convergence rates and errors of the developed DFEM for the LOQD equations we use several analytic tests formulated by the MMS.

The manufactured angular flux solution for this problem is taken to be

$$\psi_M(x,\mu) = \frac{1}{2}\phi_M(x) + \frac{3}{2}\mu J_M(x) + \frac{5}{8}\left(3\mu^2 - 1\right)\left(3E_M(x) - 1\right)\phi_M(x), \quad (46)$$

where the constants and angular dependence have been chosen such that

$$\int_{-1}^{1} \psi_M(x,\mu)\,d\mu = \phi_M(x),$$
$$\int_{-1}^{1} \mu\,\psi_M(x,\mu)\,d\mu = J_M(x),$$
$$\int_{-1}^{1} \mu^2\,\psi_M(x,\mu)\,d\mu = E_M(x)\phi_M(x).$$

We have used three values for $\phi_M(x)$, one for which the solutions are zero at the boundaries, and two for which the solutions are non-zero at the boundaries. In the former case, we take the manufactured scalar flux

$$\phi_M(x) = 8x^2(1-x) \quad \text{(MMS 1)}$$

while, in the latter case, we define

$$\phi_M(x) = 5\left[1 + x^2\left(1 - x^2\right)\right] \quad \text{(MMS 2)}$$

and

$$\phi_M(x) = 5 - \tanh\left[30\left(x - \frac{1}{2}\right)^2\right]. \quad \text{(MMS 3)}.$$

In all three cases, we use

$$E_M(x) = \frac{1}{3}\left[1 + \left(x - \frac{1}{2}\right)^2\right]. \quad (47)$$

The manufactured current is determined from the first moment equation

$$J_M(x) = -\frac{1}{\sigma_t}\frac{d}{dx}\Big(E_M(x)\phi_M(x)\Big). \quad (48)$$

These MMS problems are all solved on $x \in [0,1]$ with $\sigma_t = 1$, $\sigma_s = 1/2$, and S_8 Gauss-Legendre quadrature, for a sequence of meshes with $k = 2^n, n = 3, \ldots, 12$ mesh cells having constant mesh spacing $\Delta x = 1/k$.

The discrete L_2-norm used to compute the error measurements is

$$\|\phi_M - \phi\|_{L_2} = \left(\frac{1}{n}\sum_{i=1}^{n}\left[\bar{\phi}_{M,i} - \bar{\phi}_i\right]^2\right)^{\frac{1}{2}},$$

where n is the number of mesh cells and where $\bar{\phi}_{M,i}$ and $\bar{\phi}_i$ are the known and numerical cell-averaged scalar flux solutions,

TABLE I: $||\phi_M - \phi||_{L_2}$ for the Fully-Consistent Method.

k	MMS 1	MMS 2	MMS 3
3	1.28×10^{-3}	1.55×10^{-3}	2.39×10^{-3}
4	1.74×10^{-4}	2.13×10^{-4}	4.55×10^{-4}
5	2.27×10^{-5}	2.77×10^{-5}	6.82×10^{-5}
6	2.88×10^{-6}	3.53×10^{-6}	9.32×10^{-6}
7	3.64×10^{-7}	4.45×10^{-7}	1.22×10^{-6}
8	4.57×10^{-8}	5.59×10^{-8}	1.55×10^{-7}
9	5.72×10^{-9}	7.01×10^{-9}	1.96×10^{-8}
10	7.16×10^{-10}	8.77×10^{-10}	2.46×10^{-9}
11	8.96×10^{-11}	1.10×10^{-10}	3.08×10^{-10}
12	1.12×10^{-11}	1.37×10^{-11}	3.86×10^{-11}

TABLE II: $||\phi_M - \phi||_{L_2}$ for the Alternate method.

k	MMS 1	MMS 2	MMS 3
3	1.15	5.66×10^{-2}	9.46×10^{-2}
4	2.63	3.09×10^{-2}	8.00×10^{-2}
5	1.93×10^{-1}	1.60×10^{-2}	4.38×10^{-2}
6	5.04×10^{-2}	8.12×10^{-3}	2.25×10^{-2}
7	1.91×10^{-2}	4.09×10^{-3}	1.14×10^{-2}
8	8.32×10^{-3}	2.05×10^{-3}	5.69×10^{-3}
9	3.86×10^{-3}	1.03×10^{-3}	2.85×10^{-3}
10	1.85×10^{-3}	5.14×10^{-4}	1.43×10^{-3}
11	9.07×10^{-4}	2.57×10^{-4}	7.13×10^{-4}
12	4.44×10^{-4}	1.29×10^{-4}	3.57×10^{-4}

respectively. The L_2-norm of errors in the solution of the fully-consistent method are listed in Table I and plotted in Figure 2. These results show that the developed fully-consistent DFEM method matches the transport solution and is third-order accurate, as anticipated. The norm of errors in the solution of the low-order equations for the alternate method are presented in Table II and in Figure 3, demonstrating that the alternate method is first-order accurate.

3. Test on Convergence of Transport Iterations

We now consider a two-region transport problem with vacuum boundary conditions. The material regions are defined as follows:

1. $0 \le x \le 5$, $\sigma_t = 1$, $\sigma_s = 0.9$, $q = 1$,
2. $5 \le x \le 10$, $\sigma_t = 2$, $\sigma_s = 0.1$.

This test is used to illustrate how the QD method with the fully-consistent DFEM scheme and the standard DFEM discretization of the P_1 equations compare when used to accelerate the source iteration applied to the DFEM transport scheme. The convergence criteria is

$$\| \phi^{(s)} - \phi^{(s-1)} \|_{L_2} < \epsilon \,,$$

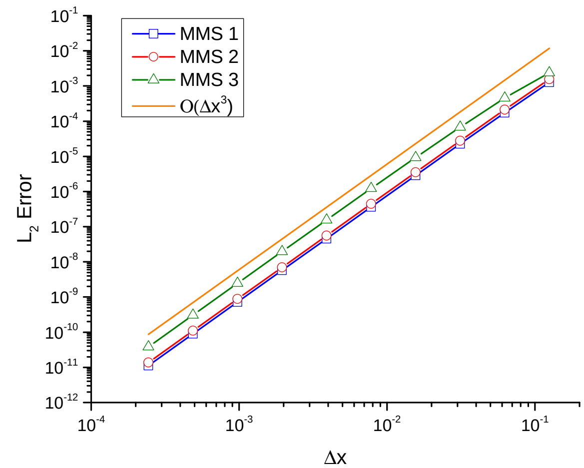

Fig. 2: Error in MMS Solutions for the Fully-Consistent Method.

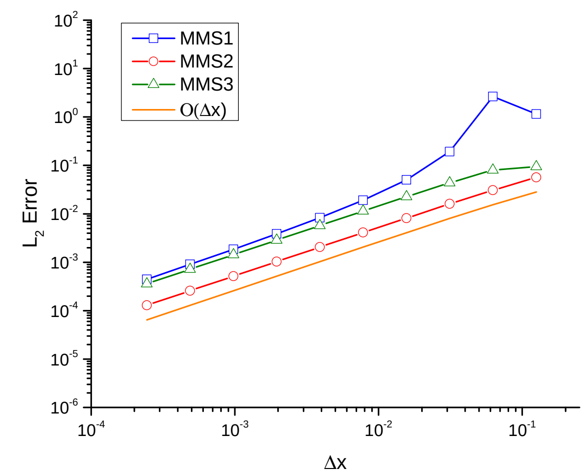

Fig. 3: Error in MMS Solutions for the Alternate Method.

where s is the iteration index and $\epsilon = 10^{-8}$. In the case of no acceleration, ϵ is scaled by

$$\frac{1-\rho}{\rho}\,,$$

where ρ is an estimate of the spectral radius. Table III shows the number of iterations required to converge an S_8 solution. We see that the proposed fully-consistent QD method efficiently accelerates transport iterations. It converges faster than the P_1SA method in problems with optically thin cells. In the case of optically thick cells, the QD method requires just one or two more iterations compared to the P_1SA method.

IV. CONCLUSIONS

We have developed the DFEM discretization of the low-order QD equations in 1D slab geometry. This scheme is fully

TABLE III: Number of iterations.

h	QD	P_1SA	SI
0.01	3	4	4
0.05	4	8	11
0.10	7	10	25
0.50	12	12	200
1.00	13	13	872
2.00	12	12	3909
5.00	13	10	14690
10.00	11	8	21574
20.00	8	7	23751
50.00	7	6	24397
100.00	6	5	24645

consistent with the DFEM approximation of the transport equation. The obtained numerical results demonstrated expected order of accuracy of the spatial discretization as well as rapid convergence of iterations. The proposed DFEM scheme for the LOQD equations can be used with various discretization schemes for the high-order transport equation. We note that another version of interface conditions can be derived on the basis of the alternative QD boundary conditions [8]. The LOQD DFEM scheme will be extended to multidimensional geometries in a future work.

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