

Neutron Diffusion and Transport Problems in Fluid-fuel Reactors

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Abstract - This paper is devoted to the physics of a homogeneous fluid-fuel system with perfect remixing. The one-group and multigroup diffusion is firstly adopted. The solution of the space integro-differential eigenvalue equation is obtained by a fully analytical method. Both multiplication and time eigenvalues are determined, and compared with the ones obtained for a standard solid fuel system. The same analytical method is then employed for the solution of the S_N problem.

I. INTRODUCTION

The molten-salt reactor is one of the concepts taken into consideration within the Generation IV undertaking [1, 2]. Furthermore fluid-core reactors are being proposed for specific applications, such as the production of radioisotopes. During the 50's Weinberg proposed the 3P system, which led to the studies at ORNL on the Homogeneous Reactor Experiment (HRE) [1, 3, 4]. At a first approach, the core can be assumed to have a spherical geometry and the fluid to be homogeneous. If the core is not neutronically very large, one can assume that the motion of the fluid determines a 'perfect mixing' situation, leading to a uniform distribution of the delayed neutron precursors inside the core.

In a previous paper [5] the neutronic one-group diffusion model of such a system was derived and investigated fully analytically. In this model the neutron balance is described by an integro-differential equation. Interesting aspects related to the structure and properties of the set of multiplication eigenvalues and corresponding eigenfunctions of the balance operator were analysed. The theory of criticality has been also developed and the results of the computation of some transients have been presented.

In this paper the main results presented in ref. [5] are briefly recalled. Afterwards, it is shown that the multiplication eigenvalues are all real and the time eigenvalue set is also analysed. Some results are presented highlighting the differences between the two eigenvalue approaches. In the following part of the work, the two-group model is discussed, presenting the analytical formulation and some numerical results for the eigenvalue spectrum. The analytical methodology used in the two-group problem can be applied to solve the problem in S_N transport, using the second-order formulation that allows to cast the problem into a set of coupled diffusion-like equations.

II. EIGENVALUE PROBLEM WITH PERFECT REMIXING - ONE-GROUP DIFFUSION

1. Multiplication k -eigenvalue problem

The problem of perfect remixing of delayed neutron precursors is tackled, as a first approach, in diffusion theory

assuming one energy group, as originally introduced in [5]. Only one family of precursors is considered for simplicity and the spherical geometry is adopted for consistency with Weinberg's proposal. The resulting eigenvalue problem for the multiplication coefficient k reads as:

$$D\nabla^2\Phi(\mathbf{r}) - \Sigma_a\Phi(\mathbf{r}) + \frac{1}{k}(1-\beta)\nu\Sigma_f\Phi(\mathbf{r}) + \frac{1}{k}\frac{\beta\nu\Sigma_f}{V}\langle\Phi\rangle = 0, \quad (1)$$

where the symbol $\langle\cdot\rangle$ denotes an integral on the spatial domain of volume V . The details on the derivation of this equation can be found also in [6].

Being the problem homogeneous, the solution to the problem is obtained first by imposing a normalization of the flux as $\langle\Phi\rangle = \Theta V$, then the equation is re-written for the unknown $U(r) = r\Phi(r)$, obtaining a source-injected diffusion equation in slab geometry, where the peculiarity is the dependence of the source on the eigenvalue:

$$\frac{d^2U_n(r)}{dr^2} - \frac{1}{L^2}U_n(r) + \frac{1}{k_n}\frac{(1-\beta)\nu\Sigma_f}{D}U_n(r) + \frac{1}{k_n}\frac{\Theta\beta\nu\Sigma_f}{D}r = 0. \quad (2)$$

The index n identifies the eigenfunction and the eigenvalue. Equation (2) is provided with boundary conditions:

$$U_n(r=0) = U_n(r=R) = 0, \quad (3)$$

where the condition in $r=0$ results from the requirement for the flux eigenfunction Φ_n to be regular in the sphere center.

The solution for the flux is:

$$\Phi_n(r) = \frac{\Theta\beta\nu\Sigma_f}{Dk_n\xi_n^2} \left(\frac{R \sin(\xi_n r)}{r \sin(\xi_n R)} - 1 \right) \equiv M_n \left(\frac{R \sin(\xi_n r)}{r \sin(\xi_n R)} - 1 \right), \quad (4)$$

where the parameter ξ , depending on the eigenvalue, is defined as:

$$\xi_n^2 = \frac{1}{k_n} \frac{(1-\beta)\nu\Sigma_f}{D} - \frac{1}{L^2} = \gamma_n - \frac{1}{L^2}. \quad (5)$$

The eigenvalue condition is obtained by imposing the normalization $\langle\Phi\rangle = \Theta V$, giving explicitly:

$$\tan(\xi_n R) = \frac{12\pi M_n \xi_n R^2}{4\pi M_n R (3 - \xi_n^2 R^2) - 3\xi_n^2 \Theta V}. \quad (6)$$

In [5] the resulting eigenvalues and corresponding eigenfunctions were discussed, focusing on the difference with respect to the standard diffusion problem (without precursor remixing). However, the solution of equation (6) requires at first to verify if the eigenvalues are real or complex.

Observing the eigenvalue equation (6) it can be easily shown that, if a complex value for the eigenvalue k exists, also its complex conjugate k^\dagger is an eigenvalue of the problem. Therefore, it is possible to prove that all k eigenvalues are real with a procedure inspired by Carslaw and Jaeger [7]. We re-write Eq. (2) in a more compact form, for two different eigenvalues γ_n and γ_m :

$$\begin{aligned} U_n'' + \left(\gamma_n - \frac{1}{L^2}\right) U_n + c\gamma_n r &= 0, \\ U_m'' + \left(\gamma_m - \frac{1}{L^2}\right) U_m + c\gamma_m r &= 0, \end{aligned} \quad (7)$$

where $c = \beta\Theta/(1 - \beta)$. We then multiply each of the two equations by the other eigenfunction

$$\begin{aligned} U_m U_n'' + \left(\gamma_n - \frac{1}{L^2}\right) U_m U_n + c\gamma_n U_m r &= 0, \\ U_n U_m'' + \left(\gamma_m - \frac{1}{L^2}\right) U_n U_m + c\gamma_m U_n r &= 0, \end{aligned} \quad (8)$$

and subtract them:

$$(U_m U_n'' - U_n U_m'') + (\gamma_n - \gamma_m) U_m U_n + c(\gamma_n U_m - \gamma_m U_n) r = 0. \quad (9)$$

Equation (9) is now integrated on the interval $[0; R]$. The first term is proven to be null, due to the boundary conditions:

$$\begin{aligned} \int_0^R (U_m U_n'' - U_n U_m'') dr &= U_m U_n' \Big|_0^R - \int_0^R U_m' U_n' dr \\ &- U_n U_m' \Big|_0^R + \int_0^R U_n' U_m' dr = 0, \end{aligned} \quad (10)$$

and the remaining terms are in the form:

$$(\gamma_n - \gamma_m) \int_0^R U_m U_n dr + c\gamma_n \int_0^R U_m dr - c\gamma_m \int_0^R U_n dr = 0. \quad (11)$$

Expression (11) can be further simplified by imposing the normalization condition on the flux, such that:

$$\int_0^R U_m dr = \int_0^R U_n dr = \frac{\Theta V}{4\pi}, \quad (12)$$

and (11) becomes:

$$(\gamma_n - \gamma_m) \left(\int_0^R U_m U_n dr + c \frac{\Theta V}{4\pi} \right) = 0. \quad (13)$$

Condition (13) needs to be fulfilled for each couple of eigenvalues; if we suppose γ_n and γ_m are complex conjugates ($\gamma_m = \gamma_n^\dagger$), also the corresponding eigenfunctions are complex conjugates, therefore the integral in (13) is strictly positive. Then, being the second term positive, equation (13) can be fulfilled only if the difference $\gamma_n - \gamma_n^\dagger$ is null, i.e. the eigenvalues are all real.

2. Time α -eigenvalue problem

The solution of the eigenvalue problem for the identification of the α -modes is also of interest for the problem of perfect remixing since, again, the nature of the equation obtained is different from usual cases. We adopt the same hypotheses as for the k -eigenvalue problem, leading to the balance equation:

$$D\nabla^2\Phi(\mathbf{r}) - \Sigma_a\Phi(\mathbf{r}) - \frac{\alpha}{v}\Phi(\mathbf{r}) + (1 - \beta)v\Sigma_f\Phi(\mathbf{r}) + \frac{\beta v\Sigma_f}{V} \langle \Phi \rangle = 0. \quad (14)$$

The imposition of the condition on the integral of the flux, together with the use of the auxiliary function $U = r\Phi$, leads to a diffusion problem as the previous one, apart for what concerns the dependence of the coefficients on the eigenvalue:

$$\frac{d^2 U_n(r)}{dr^2} + \eta_n^2 U_n(r) + \frac{\Theta\beta v\Sigma_f}{D} r = 0, \quad (15)$$

where

$$\eta_n^2 = -\frac{\alpha_n^2}{vD} - \frac{1}{L^2} + \frac{(1 - \beta)v\Sigma_f}{D}. \quad (16)$$

The solution in this case is, similarly to (4):

$$\Phi_n(r) = \frac{\Theta\beta v\Sigma_f}{D\eta_n^2} \left(\frac{R \sin(\eta_n r)}{r \sin(\eta_n R)} - 1 \right) = N_n \left(\frac{R \sin(\eta_n r)}{r \sin(\eta_n R)} - 1 \right), \quad (17)$$

and also the eigenvalue condition has the same structure as (6):

$$\tan(\eta_n R) = \frac{12\pi N_n \eta_n R^2}{4\pi N_n R (3 - \eta_n^2 R^2) - 3\eta_n^2 \Theta V}. \quad (18)$$

Although expression (6) and (18) are formally identical, their difference lies in the dependence of the coefficients on the multiplication and time eigenvalues. Therefore, it can be expected that the corresponding eigenfunctions will be different (while for standard solid fuel configurations they coincide), and the relation between k and α cannot be worked out explicitly.

Also in this case, the proof of the reality of the eigenvalues follows the lines drawn by Carslaw and Jaeger: the equations corresponding to two different eigenvalues are considered

$$\begin{aligned} U_n'' + \eta_n U_n + cr &= 0, \\ U_m'' + \eta_m U_m + cr &= 0, \end{aligned} \quad (19)$$

where $c = \beta v\Sigma_f\Theta/D$. The two equations are then combined as in the previous proof and integrated on the interval $[0; R]$:

$$\begin{aligned} \int_0^R (U_m U_n'' - U_n U_m'') dr + (\eta_n - \eta_m) \int_0^R U_n U_m dr \\ + c \int_0^R r (U_m - U_n) dr = 0. \end{aligned} \quad (20)$$

The first integral in expression (20) is null as shown in (10). The third term is null as well, due to the normalization condition imposed on both U_n and U_m . As a result the following expression is to be fulfilled:

$$(\eta_n - \eta_m) \int_0^R U_n U_m dr = 0. \quad (21)$$

Assuming η_n and η_m are complex conjugate ($\eta_m = \eta_n^\dagger$), the integral of the product of the two conjugate eigenfunctions is strictly positive and therefore $\eta_n = \eta_n^\dagger$.

3. Comparison of k and α eigenvalues and eigenfunctions

In this section some numerical results on the eigenvalues and eigenfunctions discussed above are presented. The nuclear data for these evaluations and the following cases are taken from previous analyses on molten salt reactors performed in the frame of a European research project [8] and are reported in Table I. One-group data are obtained by the two-group set by a standard energy collapsing procedure in the spherical domain considered.

TABLE I. Nuclear data in one-group and two-group adopted in the numerical evaluations.

		one-group		two-group	
		$g = 1$	$g = 1$	$g = 2$	$g = 2$
Σ_r	[cm ⁻¹]	$3.711 \cdot 10^{-1}$	$3.54 \cdot 10^{-1}$	$4.09 \cdot 10^{-1}$	
D	[cm]	$8.98 \cdot 10^{-1}$	$9.42 \cdot 10^{-1}$	$8.15 \cdot 10^{-1}$	
Σ_a	[cm ⁻¹]	$2.404 \cdot 10^{-3}$	$9.78 \cdot 10^{-4}$	$5.56 \cdot 10^{-3}$	
$\Sigma_{g \rightarrow g+1}$	[cm ⁻¹]	–	$3.08 \cdot 10^{-3}$	–	
Σ_r	[cm ⁻¹]	$2.404 \cdot 10^{-3}$	$4.058 \cdot 10^{-3}$	$5.56 \cdot 10^{-3}$	
$\nu \Sigma_f$	[cm ⁻¹]	$3.797 \cdot 10^{-3}$	0.0	$1.222 \cdot 10^{-2}$	
v	[cm/s]	$1.65 \cdot 10^6$	$1.65 \cdot 10^7$	$5.5 \cdot 10^5$	
β	[pcm]	650		650	

Table II reports the eigenvalues, evaluated for the first 10 eigenfunctions, for both eigenvalue problems. The values are compared to the corresponding k and α obtained for the standard solid fuel problem, according to the formulae:

$$k_n = \frac{\nu \Sigma_f}{\Sigma_a} \frac{1}{1 + L^2 \left(\frac{n\pi}{R} \right)^2}, \quad (22)$$

$$\alpha_n = (k_n - 1) \nu \Sigma_a \left(1 + L^2 \left(\frac{n\pi}{R} \right)^2 \right).$$

The effect on the reactivity of the system associated to the precursor remixing is clearly visible as the multiplication eigenvalue k is reduced, moving from a slightly supercritical configuration to a subcritical one. The difference is of course present for all eigenvalue index n and is visible also in terms of time-eigenvalues α_n . The corresponding eigenfunctions, differently from the standard case, are no longer the same: although the mathematical form of the two eigensolutions is the same (see expressions (4) and (17)), the coefficients couples (ξ , M) and (η , N) are slightly different, as shown in Table III. The effect on the eigenfunction shape is shown graphically in Fig. 1.

The potential use of these sets of eigenfunctions instead of the standard Helmholtz solutions has been discussed in [5], showing how the asymptotic behavior of the system can be more accurately described with the k -eigenfunctions here obtained, due to the correct evaluation of the fundamental mode of the system. Similar comments can be made as regards the potential use of the α -eigenfunctions in the presence of remixing: the stable period of the system is here obtained as the inverse of the first eigenvalue of the problem, while the adoption of standard Helmholtz eigenfunctions would result in an infinite series of exponential terms.

TABLE II. Eigenvalues k and α for the perfect remixing problem, as compared to the eigenvalue of the corresponding solid fuel diffusion problem for the first 10 eigenfunctions ($R = 80$ cm).

n	k_n		$\alpha_n \cdot 10^5$ [s ⁻¹]	
	remix	standard	remix	standard
$n = 1$	0.99945	1.00200	$-3.4242 \cdot 10^{-5}$	$1.2473 \cdot 10^{-4}$
$n = 2$	0.47526	0.47790	-0.068612	-0.068267
$n = 3$	0.25377	0.25532	-0.182632	-0.182253
$n = 4$	0.15358	0.15455	-0.342225	-0.341834
$n = 5$	0.10187	0.10252	-0.547406	-0.547010
$n = 6$	0.07217	0.07264	-0.798179	-0.797779
$n = 7$	0.05368	0.05403	-1.094545	-1.094144
$n = 8$	0.04143	0.04170	-1.436505	-1.436103
$n = 9$	0.03292	0.03313	-1.824059	-1.823656
$n = 10$	0.02677	0.02694	-2.257208	-2.256804

TABLE III. Coefficients ξ and M appearing in k -eigenfunctions, compared to the corresponding coefficients η and N in α -eigenfunctions. Same data and geometry as in the previous Table.

n	ξ_n [cm ⁻¹]	η_n [cm ⁻¹]	M_n [cm ⁻² s ⁻¹]	N_n [cm ⁻² s ⁻¹]
$n = 1$	0.0390563	0.0390564	0.0180220	0.0180120
$n = 2$	0.0784842	0.0785133	0.0093853	0.0044572
$n = 3$	0.1177789	0.1178019	0.0078048	0.0019799
$n = 4$	0.1570581	0.1570763	0.0072524	0.0011136
$n = 5$	0.1963330	0.1963478	0.0069968	0.0007127
$n = 6$	0.2356059	0.2356185	0.0068580	0.0004949
$n = 7$	0.2748779	0.2748887	0.0067743	0.0003636
$n = 8$	0.3141493	0.3141589	0.0067200	0.0002784
$n = 9$	0.3534204	0.3534289	0.0066827	0.0002200
$n = 10$	0.3926912	0.3926989	0.0066561	0.0001782

III. EIGENVALUE PROBLEM WITH PERFECT REMIXING - MULTIGROUP DIFFUSION

1. Multiplication k -eigenvalue problem

The physical problem under analysis is now generalized including the energy dependence, to show a general approach for the solution of the critical problem in multigroup diffusion, that could also be applied to transport problems in second-order form. The technique is based on a decoupling procedure leading to a set of uncoupled diffusion-like equations [9]. We start from the two-group criticality problem assuming thermal fission only:

$$\begin{cases} D_1 \nabla^2 \Phi_1(r) - \Sigma_{r1} \Phi_1(r) + \frac{1}{k} \nu \Sigma_{f2} (1 - \beta) \Phi_2(r) + \frac{1}{k} \frac{\nu \Sigma_{f1} \beta}{V} \langle \Phi_2 \rangle = 0 \\ D_2 \nabla^2 \Phi_2(r) - \Sigma_{a2} \Phi_2(r) + \Sigma_{1 \rightarrow 2} \Phi_1(r) = 0. \end{cases} \quad (23)$$

Similarly to the one-group case, the integral of the thermal flux $\langle \Phi_2 \rangle$ is set equal to ΘV . The system of equations can be

written in matrix form:

$$\begin{bmatrix} \nabla^2 & 0 \\ 0 & \nabla^2 \end{bmatrix} \begin{bmatrix} |\Phi_1\rangle \\ |\Phi_2\rangle \end{bmatrix} + \begin{bmatrix} -\frac{1}{L_1^2} & \frac{\nu\Sigma_{f2}(1-\beta)}{kD_1} \\ \frac{\Sigma_{1\rightarrow 2}}{D_2} & -\frac{1}{L_2^2} \end{bmatrix} \begin{bmatrix} |\Phi_1\rangle \\ |\Phi_2\rangle \end{bmatrix} + \begin{bmatrix} \frac{1}{k} & \frac{\Theta\beta\nu\Sigma_{f2}}{D_1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} |\Phi_1\rangle \\ |\Phi_2\rangle \end{bmatrix} \quad (24)$$

and then diagonalised by using the eigenvectors $|\psi_1\rangle$ and $|\psi_2\rangle$ of the coefficient matrix \hat{A} :

$$A = \begin{bmatrix} -\frac{1}{L_1^2} & \frac{\nu\Sigma_{f2}(1-\beta)}{kD_1} \\ \frac{\Sigma_{1\rightarrow 2}}{D_2} & -\frac{1}{L_2^2} \end{bmatrix}, \quad (25)$$

associated to the eigenvalues ϑ^2 and $-\varsigma^2$, and their adjoints $\langle\psi_1^+|$ and $\langle\psi_2^+|$, representing a bi-orthonormal basis set. The two eigenvalues are of opposite sign if the condition

$$k < (1-\beta) \frac{\nu\Sigma_{f2}}{\Sigma_{r2}} \frac{\Sigma_{1\rightarrow 2}}{\Sigma_{r1}} \rightarrow k < (1-\beta)k_\infty \quad (26)$$

is verified, as it is for the problem under consideration. The unknown flux vector is then represented as:

$$|\Phi\rangle = \begin{bmatrix} |\Phi_1\rangle \\ |\Phi_2\rangle \end{bmatrix} = f_1 |\psi_1\rangle + f_2 |\psi_2\rangle \quad (27)$$

and it is substituted into Eqs. (24). The projection on the adjoint vectors leads to two uncoupled diffusion equations for the components f_1 and f_2 :

$$\begin{cases} \nabla^2 f_1 + \vartheta^2(k)f_1 + \frac{1}{k} \frac{\nu\Sigma_{f2}\beta\Theta}{D_1} \langle\psi_1^+ | 1 \rangle_0 = \\ \nabla^2 f_1 + \vartheta^2(k)f_1 + \ell(k)\psi_{1,1}^+(k) = 0 \\ \nabla^2 f_2 - \varsigma^2(k)f_2 + \frac{1}{k} \frac{\nu\Sigma_{f2}\beta\Theta}{D_1} \langle\psi_2^+ | 1 \rangle_0 = \\ \nabla^2 f_2 - \varsigma^2(k)f_2 + \ell(k)\psi_{2,1}^+(k) = 0, \end{cases} \quad (28)$$

where $\psi_{i,j}^+$ is the j -th component of the i -th adjoint eigenvector $\langle\psi_i^+|$. Each of the above equations can be solved in spherical geometry with the same procedure as for the one-group case, introducing the auxiliary functions $U_i = rf_i$. The imposition of the regularity of the fluxes in $r = 0$ and boundary conditions in $r = R$ allows to reconstruct the two-group fluxes as:

$$\begin{cases} \Phi_1(r) = \psi_{1,1}(k) \frac{\ell(k)\psi_{1,1}^+(k)}{\vartheta(k)^2} \left(\frac{R \sin \vartheta(k)r}{r \sin \vartheta(k)R} - 1 \right) \\ \quad - \psi_{2,1}(k) \frac{\ell(k)\psi_{2,1}^+(k)}{\varsigma(k)^2} \left(\frac{R \sinh \varsigma(k)r}{r \sinh \varsigma(k)R} - 1 \right) \\ \Phi_2(r) = \psi_{1,2}(k) \frac{\ell(k)\psi_{1,1}^+(k)}{\vartheta(k)^2} \left(\frac{R \sin \vartheta(k)r}{r \sin \vartheta(k)R} - 1 \right) \\ \quad - \psi_{2,2}(k) \frac{\ell(k)\psi_{2,1}^+(k)}{\varsigma(k)^2} \left(\frac{R \sinh \varsigma(k)r}{r \sinh \varsigma(k)R} - 1 \right), \end{cases} \quad (29)$$

where $\psi_{i,j}$ is the j -th component of the i -th direct eigenvector $|\psi_i\rangle$.

The criticality condition is again retrieved by imposing the condition on the integral of the thermal flux:

$$4\pi \frac{\psi_{1,2}\psi_{1,1}^+\ell R}{\vartheta^2} \left(\frac{1}{\vartheta^2} - \frac{R}{\vartheta} \cot(\vartheta R) - \frac{V}{4\pi R} \right) - 4\pi \frac{\psi_{2,2}\psi_{2,1}^+\ell R}{\varsigma^2} \left(\frac{R}{\varsigma} \coth(\varsigma R) - \frac{1}{\varsigma^2} - \frac{V}{4\pi R} \right) = \Theta V. \quad (30)$$

The solution of Eq. (30) allows to identify an infinite set of discrete real solutions, and the numerical evaluation of the eigenvalues for a subset of harmonics is presented and discussed in the following. However, the form of such equation does not allow a simple proof of the fact that all its solutions are real (as the previous cases reported in Eqs. (6) and (18)). In addition, the approach to the reality proof as sketched in the previous session does not provide useful results in this case, leaving the mathematical question on the reality of the eigenvalue set open. Thence, one cannot exclude the presence of complex eigenvalues.

The question on the reality of the eigenvalues is especially of interest if the time-eigenvalue are considered, since it has already been shown that in multigroup diffusion the presence of complex α values is possible if at least three energy groups are considered, while two-group theory admits only real time constants [10]. However, since the physical problem under consideration shows some peculiar features with respect to the problem for solid fuel systems, the question on the presence of complex eigenvalues is legitimate, although not easy to answer.

2. Time α -eigenvalue problem

The solution of the time eigenvalue problem in two-group is readily solved as described in the previous section. Once the problem has been recast in matrix form:

$$\begin{bmatrix} \nabla^2 & 0 \\ 0 & \nabla^2 \end{bmatrix} \begin{bmatrix} |\Phi_1\rangle \\ |\Phi_2\rangle \end{bmatrix} + \begin{bmatrix} -\frac{1}{L_1^2} - \frac{\alpha}{\nu_1 D_1} & \frac{\nu\Sigma_{f2}(1-\beta)}{D_1} \\ \frac{\Sigma_{1\rightarrow 2}}{D_2} & -\frac{1}{L_2^2} - \frac{\alpha}{\nu_2 D_2} \end{bmatrix} \begin{bmatrix} |\Phi_1\rangle \\ |\Phi_2\rangle \end{bmatrix} + \begin{bmatrix} \frac{\Theta\beta\nu\Sigma_{f2}}{D_1} \\ 0 \end{bmatrix} \begin{bmatrix} |\Phi_1\rangle \\ |\Phi_2\rangle \end{bmatrix}, \quad (31)$$

the eigenvalues of the coefficient matrix, now depending on α , can be identified, together with the corresponding eigenvectors and formulae (27)-(30) are left unchanged, apart from the functional dependence of the parameters on the eigenvalue. As mentioned before, the numerical solution of Eq. (30) allows to find a set of discrete real eigenvalues, but the potential presence of complex solutions is still an open question.

In the following section numerical results for this eigenvalue problem are provided, comparing the α values with the corresponding results obtained in standard solid fuel systems. In this latter case, the time eigenvalues can be obtained, for a value $B_n^2 = (n\pi/R)^2$ of the buckling, solving the following

characteristic equation:

$$\begin{vmatrix} -\frac{1}{L_1^2} - B_n^2 - \frac{\alpha}{v_1 D_1} & \frac{v \Sigma_{f2}(1-\beta)}{D_1} \\ \frac{\Sigma_{1 \rightarrow 2}}{D_2} & -\frac{1}{L_2^2} - B_n^2 - \frac{\alpha}{v_2 D_2} \end{vmatrix} = 0, \quad (32)$$

providing the α eigenvalues, associated to the eigenfunction:

$$\begin{aligned} \phi_1(r) &= \frac{\sin(B_n r)}{r}, \\ \phi_2(r) &= \frac{\Sigma_{1 \rightarrow 2}}{D_1 \left(1 + L_2^2 B_n^2 + \alpha / (v_2 \Sigma_{a2})\right)} \frac{\sin(B_n r)}{r}. \end{aligned} \quad (33)$$

The values of α obtained in this case can be used as starting guess for the numerical procedure to identify the solutions of Eq. (30), supposing the the eigenvalue structure is preserved from the standard case to the full remixing one.

3. Evaluation of eigenvalues and eigenfuctions in two-group theory

The evaluation of both the fundamental and higher-order harmonics for the two-group problem is performed, adopting the data in Table I and $R = 80$ cm. The eigenvalues k and α up to order 8 are reported in Table IV, as compared to the results of the standard diffusion formulation for solid fuel. The reactivity loss (reduction of the fundamental k eigenvalue) associated to the remixing is coincident with the one-group case, coherently with the energy collapsing process adopted to generate the one-group data from the two-group cross sections. It can also be observed that, as expected, the higher-order k eigenvalues are much smaller than the fundamental, differently from the previous case, with well-known implications in the physical description of the system behavior [11]. In addition, the effect of remixing on higher-harmonics shows a stronger damping with increasing order. The observation of the time eigenvalues leads to similar comments.

In Figs. 2 and 3 the eigenfunctions of the k and α problem are compared to the corresponding harmonics of the standard case, showing discrepancies in the order of the percent. The peaks in the difference visible in the graphs are associated to the different localization of the zeros of the functions, together with the definition of the difference in relative terms.

The same approach can be adopted for the solution of the corresponding adjoint problem:

$$\begin{cases} D_1 \nabla^2 \Phi_1^+(r) - \Sigma_{r1} \Phi_1^+(r) + \Sigma_{1 \rightarrow 2} \Phi_2^+(r) = 0 \\ D_2 \nabla^2 \Phi_2^+(r) - \Sigma_{r2} \Phi_2^+(r) + \frac{1}{k} v \Sigma_f (1-\beta) \Phi_1^+(r) \\ + \frac{1}{k} \frac{v \Sigma_f \beta}{V} \langle \Phi_1^+ \rangle = 0, \end{cases} \quad (34)$$

using the eigenvalues and eigenvectors of the transposed coefficient matrix. The evaluation of the neutron importance is needed when adopting a perturbative approach for the study of the effect of material modifications in this kind of system, as was done in [12].

TABLE IV. Eigenvalues k and α for the perfect remixing problem in two-group diffusion theory, as compared to the eigenvalue of the corresponding standard solid fuel problem for the first 8 eigenfunctions ($R = 80$ cm).

	k_n		$\alpha_n \cdot 10^4$ [s ⁻¹]	
	remix	standard	remix	standard
$n = 1$	0.99945	1.00200	$-1.9832 \cdot 10^{-4}$	$7.2010 \cdot 10^{-4}$
$n = 2$	0.35831	0.36030	-0.368818	-0.367650
$n = 3$	0.12946	0.13025	-0.804279	-0.803529
$n = 4$	0.05339	0.05372	-1.334132	-1.333644
$n = 5$	0.02506	0.02522	-1.981410	-1.981075
$n = 6$	0.01307	0.01315	-2.756689	-2.756447
$n = 7$	0.00740	0.00745	-3.664768	-3.664586
$n = 8$	0.00448	0.00451	-4.707981	-4.707839

The method outlined here can be extended to any number of energy groups, allowing the general solution of a multi-group diffusion problem for this peculiar kind of nuclear system. Moreover, the same approach can be fruitfully exploited for the solution of the corresponding transport problem, as is done in the next section.

IV. EIGENVALUE PROBLEM WITH PERFECT REMIXING - TRANSPORT MODEL

1. One-dimensional slab model: S_N approximation

The eigenvalue problem discussed in the previous sections can be also approached in transport theory. If we adopt a monoenergetic transport model in one-dimensional slab geometry of dimension H , the resulting equation to be solved is:

$$\begin{aligned} \mu \frac{\partial \varphi(x, \mu)}{\partial x} + \Sigma \varphi(x, \mu) &= \frac{\Sigma_s}{2} \int_{-1}^1 d\mu' \varphi(x, \mu') \\ + \frac{v \Sigma_f (1-\beta)}{2k} \int_{-1}^1 d\mu' \varphi(x, \mu') &+ \frac{v \Sigma_f \beta}{2kH} \int_{-1}^1 d\mu' \langle \varphi(x, \mu') \rangle. \end{aligned} \quad (35)$$

Scattering is assumed isotropic, and the spatial integral of the flux is identified with the same notation as before. One possible approximation is based on discrete ordinates, solving for the angular fluxes in a finite set of N directions and approximating the integrals over the variable μ as quadrature sums:

$$\begin{aligned} \mu_n \frac{d\varphi(x, \mu_n)}{dx} + \Sigma \varphi(x, \mu_n) &= \left(\frac{\Sigma_s}{2} + \frac{v \Sigma_f (1-\beta)}{2k} \right) \sum_{m=1}^N w_m \varphi(x, \mu_m) \\ + \frac{v \Sigma_f \beta}{2kH} \sum_{m=1}^N w_m \langle \varphi(x, \mu_m) \rangle, \quad n &= 1, \dots, N. \end{aligned} \quad (36)$$

As customarily done, the directions and weights are taken according to the Gauss-Legendre quadrature set.

The choice of S_N allows to adopt the methodology described above for multigroup. In fact, system (36) can be rearranged by combining the equations for opposite directions (summing and subtracting them), obtaining a set of $N/2$ coupled even-parity second-order equations. As an example, the

S₄ model becomes:

$$\begin{cases} \frac{\mu_1^2}{\Sigma} \frac{\partial^2 \Phi_1}{\partial x^2} + \left(-\Sigma + \Sigma_s w_1 + \frac{(1-\beta)v\Sigma_f}{k} w_1 \right) \Phi_1 \\ + \left(\Sigma_s w_2 + \frac{(1-\beta)v\Sigma_f}{k} w_2 \right) \Phi_2 + \frac{\beta v \Sigma_f \Theta}{k} = 0 \\ \frac{\mu_2^2}{\Sigma} \frac{\partial^2 \Phi_2}{\partial x^2} + \left(\Sigma_s w_1 + \frac{(1-\beta)v\Sigma_f}{k} w_1 \right) \Phi_1 \\ + \left(-\Sigma + \Sigma_s w_2 + \frac{(1-\beta)v\Sigma_f}{k} w_2 \right) \Phi_2 + \frac{\beta v \Sigma_f \Theta}{k} = 0, \end{cases} \quad (37)$$

where the integral of the flux has been set to ΘH as before, $\mu_3 = -\mu_1$ and $\mu_2 = -\mu_4$ and the new unknowns are:

$$\begin{aligned} \Phi_1 &= \varphi(x, \mu_1) + \varphi(x, -\mu_1), \\ \Phi_2 &= \varphi(x, \mu_2) + \varphi(x, -\mu_2). \end{aligned} \quad (38)$$

System (37) is clearly in the same form as the two-group problem (24), apart from the different positioning of the k eigenvalue, and it can be solved with the same approach. A difference arises concerning the imposition of the boundary conditions: if we suppose the system symmetric around zero and impose Mark boundary conditions on the discrete ordinates fluxes [13], we obtain

$$\begin{aligned} \mu_1 \frac{d\Phi_1}{dx} \Big|_{x=H/2} + \Sigma \Phi_1(x=H/2) &= 0, \\ \mu_2 \frac{d\Phi_2}{dx} \Big|_{x=H/2} + \Sigma \Phi_2(x=H/2) &= 0, \end{aligned} \quad (39)$$

together with the request for the solution to be even. If we introduce a fictitious two-group diffusion coefficient $\tilde{D}_i = \mu_i^2 / \Sigma$, system (37) is recast in matrix form as:

$$\begin{bmatrix} \nabla^2 & 0 \\ 0 & \nabla^2 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} + \hat{A} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}, \quad (40)$$

where the coefficient matrix and source vector now are:

$$\hat{A} = \begin{bmatrix} \frac{w_1}{\tilde{D}_1} \left(-\frac{\Sigma}{w_1} + \Sigma_s + \frac{(1-\beta)v\Sigma_f}{k} \right) & \frac{w_2}{\tilde{D}_1} \left(\Sigma_s + \frac{(1-\beta)v\Sigma_f}{k} \right) \\ \frac{w_1}{\tilde{D}_2} \left(\Sigma_s + \frac{(1-\beta)v\Sigma_f}{k} \right) & \frac{w_2}{\tilde{D}_2} \left(-\frac{\Sigma}{w_2} + \Sigma_s + \frac{(1-\beta)v\Sigma_f}{k} \right) \end{bmatrix}, \quad (41)$$

$$\begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} = \begin{bmatrix} \frac{\Theta \beta v \Sigma_f}{\tilde{D}_1} \\ \frac{\Theta \beta v \Sigma_f}{\tilde{D}_2} \end{bmatrix}. \quad (42)$$

Using the same symbols as before for the eigenvalues and eigenvectors of matrix \hat{A} and solving for the auxiliary functions f_1 and f_2 , the solution for the even-parity fluxes are obtained as:

$$\begin{aligned} \Phi_1(x) &= A \cos(\vartheta x) + C \cosh(\varsigma x) \\ &\quad - \frac{\psi_{1,1}^+ \ell_1 + \psi_{1,2}^+ \ell_2}{\vartheta^2} + \frac{\psi_{2,1}^+ \ell_1 + \psi_{2,2}^+ \ell_2}{\varsigma^2} \\ \Phi_2(x) &= A \psi_{1,2} \cos(\vartheta x) + C \psi_{2,2} \cosh(\varsigma x) \\ &\quad - \psi_{1,2} \frac{\psi_{1,1}^+ \ell_1 + \psi_{1,2}^+ \ell_2}{\vartheta^2} + \psi_{2,2} \frac{\psi_{2,1}^+ \ell_1 + \psi_{2,2}^+ \ell_2}{\varsigma^2}. \end{aligned} \quad (43)$$

The imposition of the boundary conditions (39) provides a solution for the coefficients A and C , still dependent on the k eigenvalue, and the requirement on the flux integral gives the criticality condition:

$$\begin{aligned} (w_1 + w_2 \Psi_{1,2}) \left[\frac{2}{\vartheta} A \sin\left(\vartheta \frac{H}{2}\right) - \frac{\psi_{1,1}^+ \ell_1 + \psi_{1,2}^+ \ell_2}{\vartheta^2} H \right] \\ + (w_1 + w_2 \Psi_{2,2}) \left[\frac{2}{\varsigma} C \sinh\left(\varsigma \frac{H}{2}\right) + \frac{\psi_{2,1}^+ \ell_1 + \psi_{2,2}^+ \ell_2}{\varsigma^2} H \right] \\ = \Theta H \end{aligned} \quad (44)$$

In Table V the first eight eigenvalues of the problem are reported and compared to the solution of the transport problem in absence of the perfect remixing, obtained with an analogous analytical procedure. The material data adopted are the same as in all previous cases (see Table I, $H = 80$ cm). The eigenfunction shapes are then compared in Fig. 4 for the first four harmonics. The k values obtained show similar trends regarding the reactivity reduction as in previous cases, and the spatial modification of the eigenfunction is again in the range of the percent.

The problem here presented provides a significant example of how the methodologies developed in the frame of diffusion theory can be applied to S_N transport to obtain analytical solutions. However, the numerical results are not readily comparable to the previous ones, due to the different assumption regarding the geometry, since the transport problem in spherical geometry would not allow such a simple analogy with diffusion.

A possible solution in order to address also spherical geometry in transport theory is to transform the spherical transport problem into a planar case with a standard technique [14] and then apply the method here described. Numerical results in spherical geometry based on this approach will be presented in a future work.

TABLE V. Eigenvalues k for the perfect remixing problem in slab geometry solved in S₄ approximation, as compared to the eigenvalue of the corresponding standard solid fuel problem, for the first 8 eigenfunctions ($H = 80$ cm).

	remix	standard
$n = 1$	1.03484	1.03589
$n = 2$	0.27862	0.28028
$n = 3$	0.11675	0.11749
$n = 4$	0.06454	0.06496
$n = 5$	0.04169	0.04196
$n = 6$	0.02962	0.02981
$n = 7$	0.02239	0.02254
$n = 8$	0.01766	0.01777

V. CONCLUSIONS

The physics of a fluid fuel system with perfect remixing is studied in the one-group and multigroup diffusion model. The analysis is focused on the determination of the multiplication and time eigenvalues and corresponding eigenfunctions. The equation describing the system second-order differential and integral in space. The results are compared with the ones obtained for a standard solid fuel system. The analytical methods developed in diffusion theory are then applied to discrete ordinates transport. The analysis allows to gain a physical insight into the peculiar characteristics of molten salt fast reactors. In future work the analytical approach will be extended to other transport models and the nature of the eigenvalue spectrum will be investigated.

REFERENCES

1. J. A. LANE, H. G. MACPHERSON, and F. MASLAN, *Fluid Fuel Reactors*, Addison-Wesley, Reading, MA (2005).
2. L. MATHIEU, D. HEUER, E. MERLE-LUCOTTE, R. BRISSOT, C. BRUN, E. LIATARD, J.-M. L. O. MÉPLAN, A. NUTTIN, and D. LECARPENTIER, “Possible configurations for the thorium molten salt reactor and advantages of the fast nonmoderated version,” *Nuclear Science and Engineering*, **161**, 78 – 89 (2009).
3. W. C. SANGREN, *Kinetic Calculations for Homogeneous Reactors*, ORNL-1205, Oak Ridge National Laboratory, Oak Ridge, TN (1952).
4. M. W. ROSENTHAL, *An Account of Oak Ridge National Laboratory's Thirteen Nuclear Reactors*, ORNL/TM-2009/181, Oak Ridge National Laboratory, Oak Ridge, TN (2009).
5. S. DULLA, P. RAVETTO, and A. K. PRINJA, “Neutronics of fluid fuel system with perfect remixing,” in “International Conference PHYSOR2014,” Kyoto, Japan (October 2014).
6. I. PÁZSIT and A. JONSSON, “Reactor kinetics, dynamic response, and neutron noise in molten salt reactors,” *Nuclear Science and Engineering*, **167**, 61 – 76 (2011).
7. H. CARLSLAW and J. JAEGER, *Conduction of heat in solids*, Clarendon Press, Oxford, UK (1959).
8. M. DELPECH, S. DULLA, C. GARZENNE, J. KOPHAZI, J. KREPEL, C. LEBRUN, D. LECARPENTIER, F. MATTIODA, P. RAVETTO, A. RINEISKI, M. SCHIKORR, and S. M., “Benchmark of Dynamic Simulation Tools for Molten Salt Reactors,” in “International Conference GLOBAL 2003,” New Orleans (November 2003).
9. S. DULLA, *Models and Methods in the Neutronics of Molten Salt Reactors*, PhD Thesis, Politecnico di Torino, Torino, Italy (2005).
10. P. SARACCO, S. DULLA, and P. RAVETTO, “On the spectrum of the multigroup diffusion equations,” *Progress in Nuclear Energy*, **59**, 86–95 (2012).
11. S. DULLA, P. RAVETTO, M. ROSTAGNO, G. BIANCHINI, M. CARTA, and A. D’ANGELO, “Some Features of Spatial Neutron Kinetics for Multiplying Systems,” *Nuclear Science and Engineering*, **149**, 88–100 (2005).
12. S. DULLA, A. K. PRINJA, and P. RAVETTO, “Random effects on reactivity in molten salt reactors,” *Annals of Nuclear Energy*, **64**, 353–364 (2014).
13. R. CIOLINI, G. COPPA, B. MONTAGNINI, and P. RAVETTO, “Simplified P_N and A_N methods in neutron transport,” *Progress in Nuclear Energy*, **40**, 245–272 (2002).
14. B. DAVISON, *Neutron Transport Theory*, University Press, Oxford (1957).

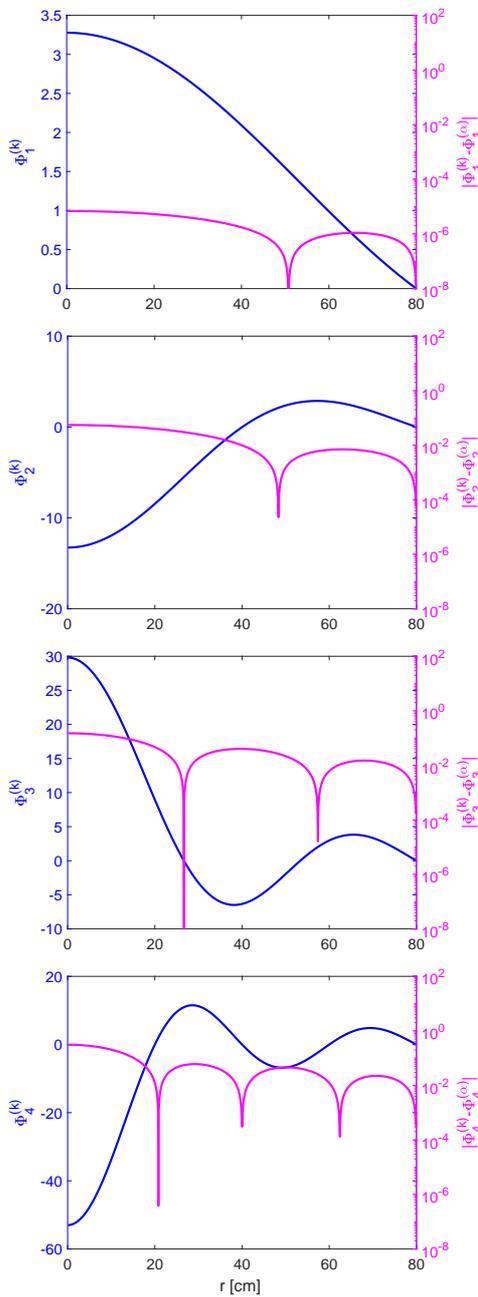


Fig. 1. Graph of the k -eigenfunction $\Phi^{(k)}$ in one-group diffusion with perfect remixing and absolute difference with respect to the α -eigenfunction $\Phi^{(\alpha)}$ for the first four harmonics. Material data from Table I, $H = 80$ cm.

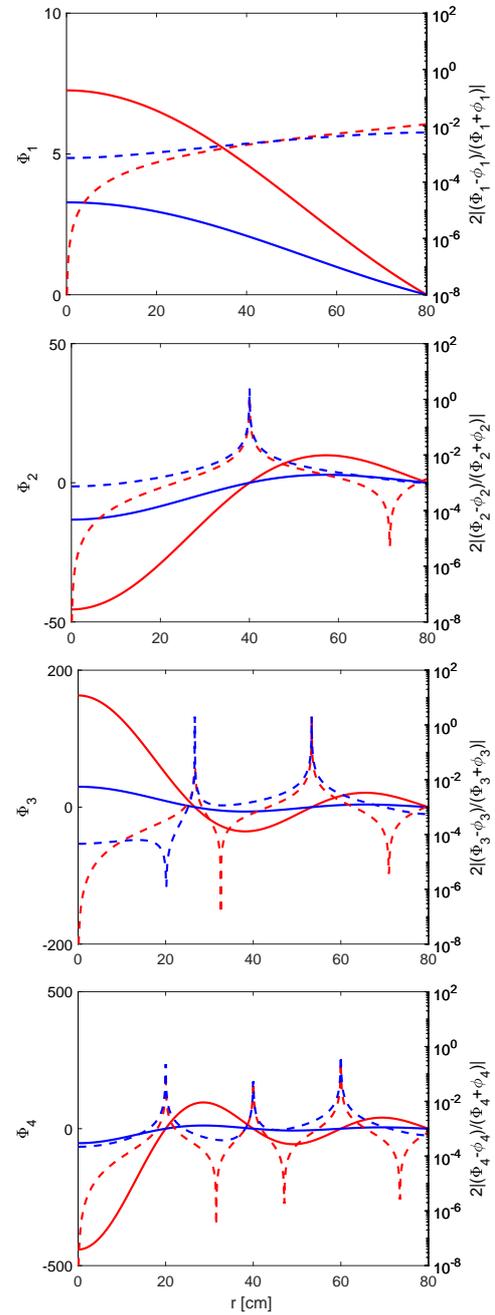


Fig. 2. Graph of the k -eigenfunction Φ (left axis, solid lines - red: fast flux, blue: thermal flux) and relative difference with respect to the k -eigenfunction of diffusion without remixing ϕ (right axis, dashed lines - red: fast flux, blue: thermal flux) for the first four harmonics. Same data and geometry as in previous graph.

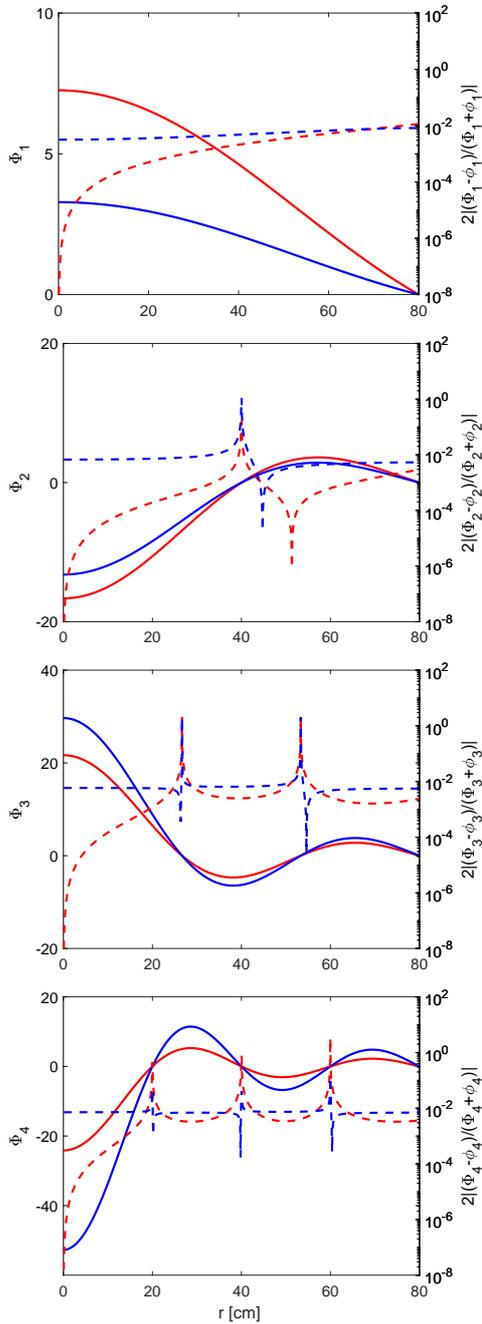


Fig. 3. Graph of the α -eigenfunction Φ (left axis, solid lines - red: fast flux, blue: thermal flux) and relative difference with respect to the α -eigenfunction of diffusion without remixing ϕ (right axis, dashed lines - red: fast flux, blue: thermal flux) for the first four harmonics. Same data and geometry as in previous graphs.

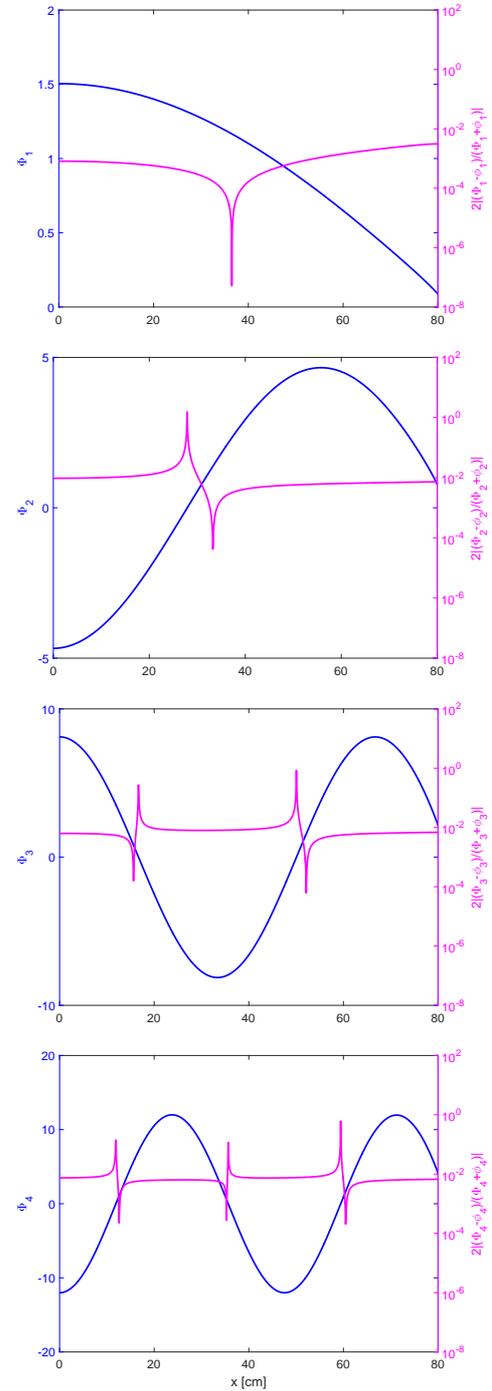


Fig. 4. Graph of the k -eigenfunction evaluated in S_4 approximation of transport with full remixing (left axis) and relative difference with respect to the k -eigenfunction of the standard solid fuel case (right axis) for the first four harmonics ($H = 80$ cm).