I. INTRODUCTION

The solution of the equation which is adjoint to the Boltzmann transport equation can be viewed as a measure of the importance of a particle to the objective function, e.g., a detector response [1]. This physical interpretation makes the adjoint flux well suited for use as a function of importance in deep penetration problems.

In this work we present a coarse-mesh method for one-speed X, Y-geometry discrete ordinates (S_N) adjoint transport problems considering isotropic scattering in non-multiplying media. In this method, the adjoint spectral Green’s function (Adjoint–SGF) scheme, originally developed for solving adjoint S_N problems in slab-geometry with non spatial truncation error [2], is applied to solve the adjoint one-dimensional transverse-integrated S_N nodal equations with constant approximation for the terms corresponding to the transverse leakages in the forward problem. A companion method has been successfully applied to X, Y-geometry forward S_N problems [3]. It is well known that the S_N equations are not only non-self-adjoint, but also have non self-adjoint boundary conditions. Nevertheless, we remark that if we use numerical methods for forward S_N problems with adjoint isotropic sources and vacuum or reflective boundary conditions, as it is applicable, we shall obtain the forward angular fluxes in the opposite directions of motion. Although this can be done in practical applications, we describe in this paper the present Adjoint–SGF–CN method, which has two new ingredients: (i) the adjoint auxiliary equation that we present in Section III.; and (ii) the adjoint partial one-node block inversion (NBI) iterative scheme, that we describe in section IV.

Numerical results to two source-detector test problems are considered in Section V. and Section VI. offers a number of general concluding remarks and suggestions for future work.

II. THE TRANSVERSE-INTTEGRATED ADJOINT S_N CONSTANT-NODAL EQUATIONS

Let us consider the one-speed adjoint S_N equations in a rectangular domain $D = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq L, 0 \leq y \leq H\}$ with isotropic scattering in non-multiplying media

\begin{align}
- \mu_m \frac{\partial \psi_n^m(x, y)}{\partial x} - \eta_m \frac{\partial \psi_n^m(x, y)}{\partial y} + \sigma_T(x, y) \psi_n^m(x, y) &
= \sum_{m=1}^{M} \psi_n^m(x, y) \sigma_n \cdot \mathbf{Q}^j(x, y) \\
&= \frac{\sigma_{S0}}{4} (x, y) \sum_{m=1}^{M} \psi_n^m(x, y) \omega_m + \mathbf{Q}^j(x, y) , \quad m = 1 : M , M = N(N+2)/2 ,
\end{align}

wherein non-outgoing adjoint flux boundary conditions apply, meaning that the importance of the leakage is clearly equal to zero, since such particles will not return to $D$, and thus, will not contribute to the detector response. In Eq (1) we have defined $\psi_n^m(x, y)$ as the adjoint angular flux in direction $\hat{\mathbf{u}}_m = (\mu_m, \eta_m)$ and $N$ is the order of the angular quadrature set. The ordered pair $(\mu_m, \eta_m)$ represents the discrete directions and $\omega_m$ are the corresponding weights. In addition, $\sigma_T$ is the total macroscopic cross section and $\sigma_{S0}$ is the isotropic scattering macroscopic cross section. The quantity $\mathbf{Q}^j(x, y)$ is the adjoint interior source, which is perfectly arbitrary [4]. Now we consider a rectangular spatial grid on $D$ where each discretization node is denoted as $d_{i,j}$, of width $h_i$ and height $h_j$. We also assume that the material parameters and the adjoint interior source are constant functions in $d_{i,j}$.

In order to obtain the one-dimensional transverse-integrated adjoint $S_N$ nodal equations, we define the generalized transverse-integration operator

\begin{align}
L_u = \frac{1}{h_x} \int_{u_{i-1/2}}^{u_{i+1/2}} du \ ,
\end{align}

where $u = x$ or $y$, and $s = i$ or $j$, respectively. First we apply $L_u$ and then $L_v$ to Eq. (1) within $d_{i,j}$ and obtain the one-dimensional transverse-integrated adjoint $S_N$ nodal equations for the $x$ and $y$ coordinate directions, respectively. That is

\begin{align}
- \frac{\mu_m}{\sigma_{T_{i,j}}} \frac{d\psi_n^m_j(x)}{dx} + \psi_{n,j}^m(x) = \frac{\epsilon_{0_{i,j}}}{4} \sum_{m=1}^{M} \frac{\psi_{n,j}^m(x) \omega_m}{\sigma_{T_{i,j}}} \\
+ \frac{1}{\sigma_{m_{ij}}} \left[ \psi_{n}^m(x, y_{j+1/2}) - \psi_{n}^m(x, y_{j-1/2}) \right] + \frac{Q^j_{i,j}}{\sigma_{T_{i,j}}}
\end{align}
and

\[- \frac{\eta_m}{\sigma_{T,i,j}} \frac{d\bar{\psi}_m^{ij}(y)}{dy} + \hat{\psi}_m^{ij}(y) = \frac{c_{0,i,j}}{4} \sum_{n=1}^{M} \bar{\psi}_n^{ij}(y) \omega_n \]

\[+ \frac{1}{\alpha_{ij}} \left[ \bar{\psi}_m^{ij}(x_{i+1/2},y) - \bar{\psi}_m^{ij}(x_{i-1/2},y) \right] + Q_{ij} \frac{1}{\sigma_{T,i,j}}, \quad (4)\]

where we have defined \( \alpha_{ij}^{\psi} \equiv \frac{h_{i+j} \sigma_{ij}}{\eta_m} \), \( \alpha_{m,i,j}^{\psi} \equiv \frac{h_{i+j} \sigma_{ij}}{\eta_m} \) and the isotropic scattering ratio \( c_{0,i,j} \equiv \sigma_{50,i,j}/\sigma_{T,i,j} \). In addition we define the averages of the adjoint angular fluxes over each spatial coordinate direction within node \( d_{i,j} \)

\[\bar{\psi}_m^{ij}(x) = \frac{1}{h_{i,j}} \int_{x_{i-1/2}}^{x_{i+1/2}} \psi_m(x,y) \, dx \quad (5)\]

and

\[\bar{\psi}_m^{ij}(y) = \frac{1}{h_{i,j}} \int_{y_{i-1/2}}^{y_{i+1/2}} \psi_m(x,y) \, dx \quad (6)\]

Equations (3) and (4) form a system of \( 2M \) ordinary differential equations in \( 6M \) unknowns. Therefore, we need to introduce approximations to guarantee uniqueness of the solution. In this paper we approximate the adjoint node-edge angular fluxes in Eqs (3) and (4) by the adjoint node-edge average angular fluxes. That is

\[\psi_m^{ij}(x,y_{j+1/2}) \approx \bar{\psi}_m^{ij}(x,y_{j+1/2}) \quad (7a)\]

and

\[\psi_m^{ij}(x_{i+1/2},y) \approx \bar{\psi}_m^{ij}(x_{i+1/2},y) \quad (7b)\]

Now, we substitute Eqs. (7a) and (7b) into Eqs. (3) and (4), respectively. For the \( x \) direction we obtain

\[- \frac{\mu_m}{\sigma_{T,i,j}} \frac{d\bar{\psi}_m^{ij}(x)}{dx} + \hat{\psi}_m^{ij}(x) = \frac{c_{0,i,j}}{4} \sum_{n=1}^{M} \bar{\psi}_n^{ij}(x) \omega_n \]

\[+ \frac{1}{\alpha_{ij}} \left[ \bar{\psi}_m^{ij}(x_{i+1/2}) - \bar{\psi}_m^{ij}(x_{i-1/2}) \right] + Q_{ij} \frac{1}{\sigma_{T,i,j}}, \quad m = 1 : M. \quad (8)\]

An analogous result is obtained for the \( y \) direction. Henceforth we shall perform our description only for the \( x \) direction, since similar results are obtained for the \( y \) direction.

At this point we solve Eq. (8) analytically, whose general solution can be written as

\[\hat{\bar{\psi}}_m^{ij}(x) = \bar{\psi}_m^{ij} + \bar{\psi}_m^{ij}(x) \quad , \quad x \in d_{i,j}\]

Here \( \bar{\psi}_m^{ij} \) is a particular solution and \( \bar{\psi}_m^{ij}(x) \) is the homogeneous component of the general solution. Substituting the spatially constant \( \bar{\psi}_m^{ij} \) into Eq. (8) we obtain

\[\bar{\psi}_m^{ij} = \frac{Q_{ij}}{\sigma_{T,j}(1 - c_{0,i,j})} + \frac{c_{0,i,j} \Delta T_j}{\sigma_{T,j} h_{i,j}(1 - c_{0,i,j})} + \bar{\psi}_m^{ij}, \quad (9)\]

where we have defined

\[\Delta T_j \equiv \frac{1}{4} \sum_{n=1}^{M} \eta_n \omega_n \left( \bar{\psi}_{n,i,j+1/2} - \bar{\psi}_{n,i,j-1/2} \right)\]

and

\[\bar{\psi}_m^{ij} \equiv \frac{1}{\alpha_{ij}} \left( \hat{\psi}_{m,i,j+1/2} - \hat{\psi}_{m,i,j-1/2} \right) \quad .\]

To determine the homogeneous component, we consider the expression

\[
\bar{\psi}_m^{ij} = a_m^{ij}(\xi) e^{-\sigma_{T,i,j}(x - \xi)} \xi, \quad a_m^{ij}(\xi) = \frac{1}{\xi} a_m^{ij}(-\xi) \quad ,
\]

(10)

where \( \delta_m,n \) is the Kronecker delta. For \( m = 1 : M \), Eq. (11) represents an eigenvalue problem. Therefore, for \( x \in d_{i,j} \) we obtain a linearly independent set of \( M \) eigenfunctions defined in Eq. (10) and we write the general solution for Eq. (8) in node \( d_{i,j} \) as

\[\bar{\psi}_m^{ij}(x) = \sum_{\xi \in d_{i,j}} \beta_k a_m^{ij}(\xi) e^{-\sigma_{T,i,j}(x - \xi)} \xi + \bar{\psi}_m^{ij} \quad .\]

Here \( a_m^{ij}(\xi) \) is the \( m \)th component of the eigenvector corresponding to the eigenvalue \( \xi_{k}^{-1} \); \( \beta_k \) are arbitrary constants and \( \bar{\psi}_m^{ij} \) is calculated by Eq. (9). Henceforth, we refer to \( \xi_k \) as an eigenvalue, although, in fact the eigenvalue is \( \xi_{k}^{-1} \), as we see in Eq. (11).

We remark at this point that the eigenvalues \( \xi_k \) for the adjacent transverse-integrated \( S_N \) equations in node \( d_{i,j} \) appear in \( \pm \) pairs due to the symmetry of the angular quadrature sets and they are all real numbers for the cases where \( c_{0,i,j} \leq 1 \). Moreover, these eigenvalues are exactly the same as the eigenvalues for the corresponding forward transverse-integrated \( S_N \) equations in node \( d_{i,j} \) [3]. On the other hand, the corresponding eigenvectors are different, but they may have the same entries located in different positions within the column matrix, as represented in Fig. 1.

Figure 1 illustrates that, if we obtain the \( M \)-dimensional eigenvectors \( a_m(\xi_k) \) corresponding to the simple eigenvalue \( \xi_k \) (multiplicity equal to one), by following a spectral analysis of the forward \( S_N \) nodal equations integrated in the \( y \) direction [3], viz Fig. 1a, then the eigenvectors \( a_m(\xi_k) \) corresponding to the same simple eigenvalue \( \xi_k \) for the adjoint \( S_N \) equations (8) interchange the first and second \( M/4 \)-dimensional arrays as well as the third and fourth \( M/4 \)-dimensional arrays, viz Fig. 1b. On the other hand, for the forward and adjoint \( S_N \) nodal equations integrated in the \( x \) direction, the eigenvectors corresponding to a given simple eigenvalue \( \xi_k \) do interchange.
the first and in third in addition to the second and fourth $M/4$–dimensional arrays. This result is due to the change of signs of the discrete ordinates $\mu_m$ and $\eta_m$.

However, for the isotropic case, we obtain $M - N$ eigenvalues $\xi_k$ equal to the values of $\mu_m$ (or $\eta_m$) according to the forward transverse–integrated $S_N$ equations in $y$ (or $x$) coordinate direction. These eigenvalues have multiplicity equal to $N - 2m + 1$ for $\mu_1 < \mu_2 < \ldots < \mu_{N/2}$. For example, if we use the $S_6$ model, we obtain one simple eigenvalue equal to $\mu_1$, three eigenvalues equal to $\mu_2$ and five eigenvalues equal to $\mu_1$. Similar results are obtained for the case considering transverse–integration in the $x$ direction, wherein the eigenvalues are equal to $\eta_m$ with multiplicity also equal to $N - 2m + 1$ for $\eta_1 < \eta_2 < \ldots < \eta_{N/2}$. Furthermore, for the adjoint transverse–integrated $S_N$ equations in $y$ (or $x$) coordinate direction, one obtains the same $M - N$ eigenvalues with the same multiplicities. However, the corresponding eigenvectors are different for the forward and adjoint problems. For the transverse integration in the $y$ direction, the eigenvectors corresponding to the same eigenvalue $\mu_m$ with multiplicity $N - 2m + 1 > 1$, $m = N/2 - 1$, may be linear combinations of the linearly independent eigenvectors corresponding to the same eigenvalue $\mu_m$, with the same interchanges as illustrated in Fig. 1.

In the next section, we derive the Adjoint–SGF–CN method that preserves the local general solution of the transverse–integrated adjoint $S_N$ nodal equations inside each node $d_{ij}$. Using boundary conditions and continuity conditions, we apply the Adjoint–SGF–CN method to solve numerically adjoint $S_N$ problems on arbitrary rectangular grids.

III. THE ADJOINT SPECTRAL GREEN’S FUNCTION CONSTANT–NODAL METHOD

Integrating Eq. (1) within an arbitrary spatial cell $d_{ij}$ by using the operator

$$\frac{1}{h_{xi} h_{yj}} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \cdots \ dy \ dx,$$

we obtain the discretized adjoint spatial balance $S_N$ equations

$$- \frac{4\mu_m}{h_{xi}} \Delta \tilde{\Psi}_{m,i} - \frac{4\mu_n}{h_{yj}} \Delta \tilde{\Psi}_{m,j} + 4 \sigma_{T_{ij}} \tilde{\Psi}_{m,j} = \sigma_{S_{0|ij}} \sum_{n=1}^{M} \tilde{\Psi}_{n_{ij}} \omega_n + 4 \tilde{Q}_{ij},$$

(13)

where we have defined

$$\Delta \tilde{\Psi}_{m,i} \equiv \tilde{\Psi}_{m,i+1/2,j} - \tilde{\Psi}_{m,i-1/2,j}, \ \Delta \tilde{\Psi}_{m,j} \equiv \tilde{\Psi}_{m,i,j+1/2} - \tilde{\Psi}_{m,i,j-1/2}$$

and the node–average adjoint angular flux in cell $\Omega_j$

$$\tilde{\Psi}_{m,j} = \frac{1}{h_{xj} h_{yi}} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \tilde{\Psi}_{m}(x,y) \ dy \ dx.$$

Equation (13), within an arbitrary spatial cell $d_{ij}$, represents a system of $M$ algebraic linear equations in $3M$ unknowns. Therefore, we need to use $2M$ auxiliary equations. In the Adjoint–SGF–CN method we use an auxiliary equation of the form

$$\tilde{\Psi}_{m,i,j} = \sum_{\mu_n < 0} \Lambda_{ij}^{\mu_n} \tilde{\Psi}_{n_{ij,j}} + \sum_{\mu_n > 0} \Lambda_{ij}^{\mu_n} \tilde{\Psi}_{n_{ij,j}} + \tilde{G}_{m,i,j}(Q_{ij}) ,$$

(15)

where $\tilde{G}_{m,i,j}(Q_{ij})$ is a function of the interior adjoint source to be determined such that the particular solution is automatically preserved. To determine the term $\tilde{G}_{m,i,j}(Q_{ij})$ we substitute Eq. (9) into Eq. (15) and the result appears as

$$\tilde{G}_{m,i,j}(Q_{ij}) = \sum_{n=1}^{N} \delta_{m,n} - \Lambda_{ij}^{\mu_n} \tilde{\Psi}_{m,j}.$$

We determine the parameters $\Lambda_{ij}^{\mu_n}$ by requiring that the homogeneous component of the local general solution be preserved by using equation (10) in equation (15) and, after some algebraic manipulations, we obtain the following linear systems:

$$\hat{\xi}_k a_m(\hat{\xi}_k) \left(1 - e^{-\frac{\sigma_{T_{ij}} h_{xi}}{\mu_m}}\right) = \sum_{\mu_n < 0} a_m(\hat{\xi}_k) \Lambda_{ij}^{\mu_n} + \sum_{\mu_n > 0} a_m(\hat{\xi}_k) \Lambda_{ij}^{\mu_n} , \ (\hat{\xi}_k > 0)$$

(16)

and

$$|\hat{\xi}_k| a_m(\hat{\xi}_k) \left(1 - e^{-\frac{\sigma_{T_{ij}} h_{xi}}{\mu_m}}\right) = \sum_{\mu_n < 0} a_m(\hat{\xi}_k) \Lambda_{ij}^{\mu_n} + \sum_{\mu_n > 0} a_m(\hat{\xi}_k) \Lambda_{ij}^{\mu_n} , \ (\hat{\xi}_k < 0).$$

(17)

Requiring this to hold for $m = 1 : M$ and $k = 1 : M$, we obtain a linear system of $M^2$ equations in the $M^2$ unknowns $\Lambda_{ij}^{\mu_n}$.

We note that the present adjoint auxiliary equation (15) and the analogous one for the $y$ direction are very similar to the auxiliary equation used for the forward SGF–CN method [3]. However, in the present Adjoint–SGF–CN method, the adjoint node–average angular flux in direction ($\mu_m$, $\eta_m$) is related to the adjoint node–average angular fluxes in all outward directions and interior adjoint source, as opposed to the inward directions for the forward $S_N$ problems. Therefore, the parameters $\Lambda_{ij}^{\mu_n}$ can be viewed as the weights of the contributions of the outgoing particles through the node–edges, which have importances $\psi_{n_{ij,j}}$ to the node–average importance of particles travelling in direction ($\mu_m$, $\eta_m$), given by $\tilde{\Psi}_{m,i,j}$, for the detector response, that depends on the numerical value attributed to the adjoint source $Q_{ij}$, normally equal to the macroscopic absorption cross section. Clearly the outgoing particles have non–zero importance to the detector response, since they might come back to node $d_{ij}$, except if they leave the domain, as we have mentioned in Section II.
IV. ADJOINT PARTIAL ONE-NODE BLOCK INVERSION (NBI) ITERATIVE SCHEME

In order to generate numerical solutions of the Adjoint–SGF–CN equations, we use the adjoint partial one–node block inversion (NBI) iterative scheme. This iterative scheme uses the most recent available estimates for the outgoing adjoint node–edge average angular fluxes (solid arrows in Fig. 2), to solve the resulting adjoint $S_N$ problem in that cell for all the incoming adjoint node–edge average angular fluxes in the opposite sweeping direction (dashed arrows in Fig. 2), which constitute the outgoing adjoint node–edge average angular fluxes for the adjacent cell in the sweeping direction. Each arrow in Fig. 2 represents $N(N+2)/8$ directions in each quadrant for the $(1-3)$ sweeping direction.

The algorithm is based on iterating on the node–edge average angular quantities in four sweeping directions: $(1-3)$, $(3-1)$, $(2-4)$ and $(4-2)$. Therefore, we substitute the Adjoint–SGF auxiliary equation (15) and the corresponding for the y coordinate direction into the balance equation (13) to remove the node–average angular quantities.

In the following section, we present numerical results illustrating the performance of the Adjoint–SGF–CN method with the adjoint partial NBI iterative scheme.

V. NUMERICAL RESULTS

In this section we consider two model problems. The first model problem consists of a uniform isotropic neutron source ($Q_1$) surrounded by a shielding material ($Q_2 = 0$). Figure 3 represents one–fourth of the whole shielding structure. The second model problem is a typical well–logging problem results were obtained by using a computational tool developed with the programming language C++ (Embarcadero® C++ Builder XE2).

1. Model Problem N² 1

Let us consider a uniform isotropic neutron source ($Q_1 = 1\ \text{cm}^{-3}\ \text{s}^{-1}$) surrounded by a shielding material ($Q_2 = 0$) as represented in Fig. 3. Here we consider reflective boundary conditions for both left and bottom boundaries and prescribed boundary conditions for both right and upper boundaries.

In this work we simulate the detection of neutrons by two identical detectors $D_1$ and $D_2$ ($\sigma_A = 1.9\ \text{cm}^{-1}$) located as illustrated in Fig. 3. To solve the adjoint problem, we set the adjoint sources numerically equal to the detector absorption macroscopic cross section ($\sigma_A$), i.e., $Q_1' = 1$, and ran the two adjoint problems using the offered Adjoint–SGF–CN method with $D_1$ and $D_2$ separately. Assuming prescribed boundary conditions for the forward $S_N$ problem and zero outgoing adjoint flux boundary conditions for the adjoint $S_N$ problem at the top and right boundaries of Fig. 3, the detector response can be obtained by

$$R = \langle \psi', Q \rangle + \int_I d\Gamma \int_{\Gamma} d\Omega \int_{\Omega} d\hat{\Omega} \psi^\dagger \left( \hat{r}, \Omega \right) \psi \left( r, \hat{\Omega} \right) .$$

where we have defined the integral operation [7]

$$\langle \cdot, \cdot \rangle \equiv \int_I d\Gamma \int_{\Gamma} d\Omega .$$

Here $\Gamma$ is the contour surface and $\Omega$ is the direction of motion.

We used the $LQ_{16}$ angular quadrature set [7] and two spatial grids. The results for the forward problem were obtained by using the forward SGF–CN method [3].

![Fig. 3: Geometry and nuclear data for Model Problem N² 1.](image-url)
of the adjoint technique to calculate the detector response is convenient as it is possible to use the same adjoint solution for various distinct intensities and/or locations of the interior sources and/or type of boundary conditions, provided we do not change the location and the type of the detector.

<table>
<thead>
<tr>
<th>Spatial grid: Number of nodes in the x direction × number of nodes in the y direction.</th>
<th>Neutron detection for the Model Problem N=2 1.</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 × 20</td>
<td>R1 (cm⁻¹ s⁻¹)</td>
</tr>
<tr>
<td>Forward</td>
<td>{1; 1; 2}</td>
</tr>
<tr>
<td>Adjoint</td>
<td>{1; 0; 0}</td>
</tr>
<tr>
<td></td>
<td>{0; 1; 2}</td>
</tr>
<tr>
<td></td>
<td>{1; 1; 2}</td>
</tr>
<tr>
<td>40 × 40</td>
<td>R1 (cm⁻¹ s⁻¹)</td>
</tr>
<tr>
<td>Forward</td>
<td>{1; 1; 2}</td>
</tr>
<tr>
<td>Adjoint</td>
<td>{1; 0; 0}</td>
</tr>
<tr>
<td></td>
<td>{0; 1; 2}</td>
</tr>
<tr>
<td></td>
<td>{1; 1; 2}</td>
</tr>
</tbody>
</table>

Absorption rate density per unit length.

{Q; BCT; BCR}: numerical values of the interior source Q and the isotropic boundary conditions on the top and the right-hand side boundaries, viz. Fig. 3.

TABLE I: Neutron detection for the Model Problem N=2 1.

Figures 4 and 5 shows the distributions of the importance functions of neutrons from an adjoint source at the detector position D1 and D2, respectively. Clearly, high importance values appear near the adjoint sources.

Fig. 4: Importance function distribution for the Model Problem N=2 1. (adjoint source D1).

Fig. 5: Importance function distribution for the Model Problem N=2 1. (adjoint source D2).

2. Model Problem N=2 2

The second model problem represents an oil well–logging problem for geophysics applications. This model problem was considered in [5] and [6], where one numerical experiment consisted in calculating the average scalar flux in region D1 due to an isotropic unit source located in region Q1 (Fig. 6). The geometry and nuclear data for this test problem are shown in Fig. 6.

In this model problem we consider four numerical experiments to illustrate the efficiency of the adjoint technique for the source–detector transport calculations. To solve the adjoint problem we used the present Adjoint–SGF–CN method on spatial grids composed of 56×64 and 112×128 nodes with the level symmetric S6 and S16 angular quadrature sets [7]. For these experiments we also set the adjoint source numerically equal to the detector absorption macroscopic cross section; i.e., Q1 = 0.004662 and ran the adjoint problem considering reflective boundary conditions at the bottom and zero outgoing adjoint angular fluxes at the other three boundaries [7]. Table II displays the numerical results for the D1 detector response due to a unit source Q1, and then Q2, and then Q3 independently and all together. We remark that we ran the Adjoint–SGF–CN code only once and then used the adjoint numerical solution in Eq. (18) to evaluate the detector response for each experiment. This was possible because the detector was not changed nor displaced.

On the other hand, if we were to run the forward S_N problems, we should run four times the forward SGF–CN code, one for each location of the unit source in Fig. 6 and one for all the three sources together, apart from the fact that the latter is just the sum of the former three detector readings. As we see, the results are very accurate with respect to the forward calculations. Figure 7 shows the importance function distribution for this model problem. As we see, the particles
TABLE II: Neutron detection $R\ (cm^{-1}\ s^{-1})$ for Model Problem N2.

<table>
<thead>
<tr>
<th>Spatial grid: Number of nodes in the x direction $\times$ number of nodes in the y direction.</th>
<th>$S_6$</th>
<th>$S_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forward</td>
<td>$112 \times 128$</td>
<td>4.31868 $\times 10^{-2}$</td>
</tr>
<tr>
<td>Adjoint</td>
<td>$112 \times 128$</td>
<td>3.01619 $\times 10^{-2}$</td>
</tr>
<tr>
<td>${Q_1; Q_2; Q_3}$: numerical values of the interior sources $Q_1$, $Q_2$, and $Q_3$, viz. Fig. 6.</td>
<td>9.33200 $\times 10^{-3}$</td>
<td>9.35334 $\times 10^{-3}$</td>
</tr>
<tr>
<td>(adjoint source $D_1$).</td>
<td>3.69181 $\times 10^{-3}$</td>
<td>3.70093 $\times 10^{-3}$</td>
</tr>
<tr>
<td>(adjoint source $D_2$).</td>
<td>4.31869 $\times 10^{-2}$</td>
<td>4.32748 $\times 10^{-2}$</td>
</tr>
</tbody>
</table>

According to the model problems considered in the previous section, the numerical results for the detector response, as generated with forward and adjoint techniques, were identical up to the sixth decimal place for the first problem and up to the fifth decimal place for the second problem. We note that the use of the adjoint technique to calculate the detector response is convenient as it is possible to run the adjoint problem just once, provided we do not change the location nor the type of the detector. We stress at this point that, even though we can generate adjoint $S_N$ solution artificially by use of the forward $S_N$ numerical methods, we offer in this paper an accurate nodal method for coarse-mesh, one-speed, adjoint $S_N$ calculations in $X, Y$-geometry with isotropic scattering.

We intend to apply these ideas to an arbitrary order $L$ of scattering anisotropy and energy multigroup $S_N$ problems in $X, Y$-geometry to account for the energy transfer in scattering events.

**VII. ACKNOWLEDGMENTS**

The authors acknowledge the financial support of the National Institute of Science and Technology on Innovative Nuclear Reactors, Brazil, for the ongoing development of this work. The work by Jesús Pérez Curbelo was supported by Fundação Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES–Brazil) and Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPEJR–Brazil).

**REFERENCES**