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On the Optimal Control in the Linear Time Invariant System with non Terminal Boundary Conditions

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Abstract

The linear sequence method is expanded in such a way that it may be applied to the boundary problem with non terminal state condition and its possibility under the existence of a corresponding costate vector $P^*(t)$ is found. For an application a couple of the concrete physical models are illuslrated and examined the effect of the sequence.

요 약

선형 수차 방법을 최종 상태가 정해져 있지 않은 경계치 문제에 확장할 수 있겠금 시도해 본 것이다. 상응 Costate vector가 존재한다는 필요 조건으로 풀 수 있음을 밝혀보았다.

응용으로써 몇개의 구체적인 물리 Model가 예로 들어졌다. 그리고 이 수차 방법의 효과가 검토되었다.

1. Introduction

One of the powerful means of obtaining the optimal condition in optimal design and the optimal control function in optimal control is Pontryagin's maximum principle. To utilize this principle, however, one must solve the two point boundary value problem. Various approaches toward solving the problem have been introduced by many people[1-4]. The author, too, has designed an approach which is capable of finding the optimal control function $u^*(t)$ by obtaining the optimal costate initial condition vector π^* through the sequence method[5]. In the cases studied, all the terminal conditions were fixed. The boundary problem with non teriminal state conditions which the author is going to introduce was not widely discussed in these papers.

In this paper the author has attempted to expand the optimal theory pertaining to the linear model, which he had introduced in an ealier paper[5], so that the theory may deal with the linear boundary problem with no terminal state conditions. The author has found that the new method is capable of solving the problem under an assumption which the corresponding costate vector $P^*(t)$ exists.

In section 2, the problem is analyzed theoreticaly. The adaptation of the sequence method for obtaing the optimal control is introduced in section 3, and its connections with previous results are pointed out. In section 4, the condition for convergence of the sequence is investigated. The physical significance of the problem is analyzed in section 5, while section 6 supplies a couple of illustrative examples as an application. Finally section 7 is left for conclusion.

2. Theoretical Analysis

The linear fixed time invariant system to the state with non torminal boundary conditions is examined. The theoretical analysis of the optimal control method for the problem is developed in such a way that the problem is reduced to a two-point boundary value problem, changed to integral form, replaced by a sequence of appoximate integral equations, and made ready for the numerical solution by the application of the sequence method[5].

We shall consider a system which is described by the following differential equation.

a. The state space approach to continuous linear control problems begins by writing the system equation in the usual way, as

$$\dot{x}(t)=Ax(t)+bu(t) \tag{1}$$

where A, b are constant matrices having dimensions n x n and n x r, respectively. Then the vector $x(t)$ is the state and the r vector $u(t)$ is the control.

b. A fixed time interval

$$t\epsilon[0,\ t_1] \tag{2}$$

c, Initial and terminal boundary conditions on the state vector

$$x(o)=\xi$$
$$x(t_1)=\text{free} \tag{3}$$

d. The control variable must satisfy a constraint

$$|u(t)|\leq| \text{ for all } t\epsilon[o,t_1] \tag{4}$$

e. The cost functional is

$$J(u)=\int_0^{t_1}|u(t)|dt. \tag{5}$$

Then, it is desired to find a control variable $u^*(t)$ that

a. Satisfies the constraint (4)

b. Transfers the system (1) from the initial state at time $t=0$ to the uncertain terminal state at time $t=t_1$

c. Minimizes the cost functional (5)

The relations deduced by applying Pontryagin's minimum principle to the problem are summarized below.

Definition 1; The "deadzone" function, dez [—], is defined as follows;

$$u(t)=\text{dez }[w(t)] \tag{6}$$

means $u(t)=1$ when $w(t)>1$
$u(t)=0$ when $|w(t)|<1$
$u(t)=-1$ when $w(t)<-1$.

Let $H(x,u,p,t)$ denote the real-valued function of the n vector x, the n vector p, and the m vector u given by

$$H(x,u,p,t,)=L(x,u,t,)+\{p,f(x,u,t)\}$$

where $f(x,u,t)$ is the function which determines the system and $L(x,u,t)$ is the intergrand of cost functional. $H(x,u,p,t)$ is called the Hamiltonian function of the problem and that p is a costate vector.

Let $u^*(t)$, $t\epsilon[o,t_1]$ be the optimal control, the solution of problem assuming that one exists. Let $x^*(t)$ be the resulting state on the optimal trajectory. In order that $u^*(t)$ be optimal, it is necessary that there exist a corresponding costate vector. Let $P^*(t)$ $t\epsilon[o,t_1]$ be the corresponding costate vector.

Then, the minimum principle yields the relations such that;

a. Hamiltonian function

$$H(x^*u^*,p^*,t)$$
$$=|u^*(t)|+p^{*\prime}(t)Ax^*(t)+p^{*\prime}(t)bu^*(t) \tag{7}$$

b. $P^*(t)$ corresponds to $u^*(t)$ and $x^*(t)$, so that $P^*(t)$ and $x^*(t)$ are a solution of the canonical system (differential equation).

$$x(t)=\frac{\partial H}{\partial p^*}=Ax^*(t)+bu^*(t) \tag{8}$$

$$p^*(t) = -\frac{\partial H}{\partial x^*} = -A'p^*(t) \tag{9}$$

where A' is the transpose of A.

c. Satisfying the boundary conditions

$$x^*(o) = \xi$$
$$x^*(t_1) : \text{free} \tag{10}$$

d. The function $H(x^*, u^*, p^*, t)$ has an absolute minimum as a function of u over Ω at $u = u^*(t)$ for in $[o, t_1]$; that is,

$$\min_{u \in \Omega} H(x^*, u, p^*, t) = H(x^*, u^*, p^*, t)$$

or, equivalently

$$H(x^*, u^*, p^*, t) \leq H(x^*, u, p^*, t) \text{ for all } u \text{ } in \Omega$$

and the optimal control $u^*(t)$ which absolutely minimizes the Hamiltonian for all u such that $|u| \leq 1$ is given by

$$u^*(t) = -\text{dez } [b'P^*(t)] \tag{11}$$

from above Eqs. (8)~(11), determination of π^*, the optimal costate initial condition vector, will be considered equivalent to the solution of the two point boundary value problem.

If any of the state variables are not fixed at the given final time, then the corresponding costate variable is fixed at time $t = t_1$ as a consequence of Pontryagin's maximum principle[11]. This reduces the number of variable to be found. Let the first r components of the state vector be fixed at time $t = T$, and define the r vector

$$\theta_j = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_r \end{pmatrix} \tag{12}$$

Define the costate final condition vector π_f and partition it.

$$\pi_f = \begin{pmatrix} \pi_j \\ \cdots\cdots \\ \pi_l \end{pmatrix} \tag{13}$$

where π_l is now the known final boundary condition on the costate[8) 13)]. The costate boundary condition on π_l is given in general form by

$$\pi_l = \frac{\partial J(x, u, t)}{\partial x_l(T)}$$

where

$$x_l = \begin{pmatrix} x_{r+1} \\ \vdots \\ x_n \end{pmatrix}.$$

Also, the foundamental matrix must be partitioned

$$e^{At} = \begin{pmatrix} e_{jj} & \vdots & e_{jl} \\ \cdots\cdots & \vdots & \cdots\cdots \\ e_{lj} & \vdots & e_{ll} \end{pmatrix},$$

Note: in each of these definitions it is understood that

$$1 \leq j \leq r \text{ and } r+1 \leq l \leq n.$$

First write the solution of Eq. (9)

$$P^*(t) = e^{-A't} \pi^* \text{ where } \pi^* = P^*(o),$$

Define for convenience

$$q(t) = e^{-At} b \tag{14}$$

Then, the vector $q(t)$ is changed to account for the final costate vector π_f. Let

$$q_f(t) = e^{-A(t-T)} b = e^{At} q(t) \tag{15}$$

and

$$q_f(t) = \begin{pmatrix} q_j(t) \\ \cdots\cdots \\ q_l(t) \end{pmatrix}.$$

Then the optimal control function (11) becomes

$$\begin{aligned} u^*(t) &= -\text{dez}[b'e^{-A't}q'(t)\pi_f] \\ &= -\text{dez}[q'_f(t)\pi_f] \\ &= -\text{dez}[q'_j(t)\pi_j + q'_l(t)\pi_l]. \end{aligned} \tag{16}$$

The solution for the state Eq. (8) is

$$x^*(t) = e^{At}[\xi - \int_o^t q(t) \text{ dez}[q'_f(t)\pi_f]dt] \tag{17}$$

and the operator $T(\pi_j)$ is now defined on the space R_r

$$T(\pi_j) = -\theta_j + [e_{jj} : e_{jl}]$$
$$[\xi - \int_o^T q(t)\text{dez}[q'_j(t)\pi_j + q'_l(t)\pi_l]dt] \tag{18}$$

3. Approach for Controllability.

A sequence of approximate operator $\{T_k(\pi_j)\}$ is now introduced to replace the operator $T(\pi_j)$. The idea is to start with a very simple operator and to work up by step toward the exact operator $T(\pi_j)$. By doing this properly, the sequence method can be guaranteed to coverge at each step, so that a workable computational approach results. Two approximations will be considered. One is a linear term to get the computations started sucessfully, and the other is a sequence of smooth function $U_k(\cdot)$ with parameter $\eta_k(k=0, \cdots\cdots i)$ as $\eta_k \to \infty$, $U_k(\cdot) \to u^*(\cdot)$, so the idea is to start with a linear approximation, then to drive the linear part to zero and increase η_k so that the approximate control $U_k(\cdot)$ converges to the optimal control $u^*(\cdot)$ when the optimal control $u^*(q'_f(t)\pi_f)$ is replaced by u_k, the form of the optimal control argument $q_f(t)\pi_f$ will be retained.

Now a linear one for the simplest useful control will be started.

Change 1; First apply a linear control using the control largument $q'_f(t)\pi_f$ yields

$$u_0(q'_f(t)\pi_f) = \alpha_0 q'_f(t)\pi_f$$

Inserting this control into the differential Eq. (1) and applying the given boundary condition leads to the zeroth approximate operator

$$T_0(\pi_j) = \xi - \int_0^T q'(t)\alpha_0 q'_f(t)\pi_f dt.$$

Let $W_f(T)$ be the controllability matrix

$$W_f(T) = \int_0^T q(t)q'_f(t)dt \tag{19}$$

Then $T_0(\pi_f) = \xi - \alpha_0 W_f(T)\pi_f$.

For approximation of the optimal control function, the exponential form $u_k(\cdot)$ can be brought in by a scalar approximation factor η_k. The deadzone function can also be approximated as closely as desired by an analytic function, since the points of discontinuity are excluded.

Change 2; Introduced an approximate control function $u_k(\cdot)$, using the control argument $q'_f(t)\pi_f$ yields

$$U_k(q'_f(t)\pi_f) = \frac{1}{2}\{\tan h[\eta_k(q'_f(t)\pi_f+1)] + \tan h[\eta_k(q'_f(t)\pi_f-1)]\}. \tag{20}$$

The general deadzone function can be replaced by an approximate function u_k and the corresponding approximate operator $T_k(\pi_j)$ from the above changes,

$$T_k(\pi_j) = -\theta_j + [e_{jj} \vdots e_{ji}][\xi - \alpha_k W_f(T)\pi_f - \int_0^T q(t)u_k[q'_f(t)\pi_f]dt]. \tag{21}$$

Let the sequence of approximate operator have k steps

$$0 < \eta_1 < \eta_2 \cdots\cdots\cdots\cdots < \eta_{k1} < \infty$$

$$\alpha_0 > \cdots\cdots\cdots\cdots > \alpha_{k2} \triangleright \alpha_{k2+1} >= \alpha_{k1} = 0 \tag{22}$$

where $k_2 < k_1$.

Definitition 2; Applied sequentially means the solution vector π_{k-1} of the previous operator $T_{k-1}(\pi_j)$ is used as a starting vector for sequence method on the present operator $T_k(\pi_j)$.

Properties of the sequence are briefly listed here according to the analytical result.

1. A sequence can be found such that sequence method converges when applied to each member sequentially.
2. Under suitable restriction this sequence of operator converges to exact operator $T(\pi_j)$.
3. The solution to the approximate operators leads to sub-optimal control which use only a little more fuel than the optimal control, yet do not require the instantaneous switching of the optimal control.

Definition 3; Assume the solution vector π_j of the operator $T_k(\pi_j)$ has been found. Now make changes $\Delta\eta$ and $\Delta\alpha$ in parameters η and α to form a new operator T_{k+1}. Applying sequence method to T_{k+1} sequentially. The set of all changes $\Delta\eta$ and $\Delta\alpha$ such that sequence method converges is called the region of convergence about η_k and α_k in the parameter space. There is a corresponding region of convergence in the space π_f of solution vectors. The region of convergence soon includes the exact solution vector π_j.

The sequence method is to be applied to a typical operator $T_k(\pi_j)$. Given the operator Eq.(21) to find the solution vector π_j such that:

$$T_k(\pi_j)=0. \tag{23}$$

One linearizes about the current guess π^i

$$T_k(\pi_j)\approx T_k(\pi_j^i)+(\pi_j-\pi_j^i)T_k^{(1)}(\pi_j^i),$$

Then the next iteration is found by solving the linear equation for π_f and the recursion relation of the sequence method is

$$\pi_j^{i+1}=\pi_j^i-[T_k^{(1)}(\pi_j^i)^{-1}T_k(\pi_j^i). \tag{24}$$

Since T_k has a vector valued in the range space, its first derivative operator is an r x r matrix

$$T_k^{(1)}(\pi_j)=-[e_{jj}:e_{jl}][\alpha_k W_f(T)+\int_0^T q(t)q'_f(t)u_k[q'_f(t)\pi_j+q'_l(t)\pi_l]dt]. \tag{25}$$

Then Eq. (23) can be written out entirely in matrix notation by substituting Eqs. (18), (21) and (25).

$$\pi_j^{i+1}=\pi_j^i-[-[e_{jj}:e_{jl}][\alpha_k\int_0^T q(t)q'_f(t)dt+\int_0^T q(t)q'_j(t)u_k^{(1)}[q'_j(t)\pi_j+q'_l(t)\pi_l]dt]^{-1}[-\theta_j+[e_{jj}:e_{jl}][\xi-\alpha_k\int_0^T q(t)q'_f(t)dt\pi_f-\int_0^T q(t)u_k[q'_f(t)\pi_f]dt]]. \tag{26}$$

From Eq. (19), the first derivative of the approximate control function $u_k^{(1)}$ is

$$u_k^{(1)}[q'_f(t)\pi_f]=\frac{1}{2}\eta_k\{2-\tanh^2[\eta_k(q'_f(t)\pi_f+1)]-\tanh^2[\eta_k(q'_f(t)\pi_f-1)]\}. \tag{27}$$

Starting with an initial guess and assuming $\pi_j^i\approx\pi_j^{*-1}$ at the same step i, the repeated application of the Eq. (20) gives the result that the inner loop is said to have converged and the vector π_j^i is defined to be the solution vector π_j of the operator T_k.

If none of the state variables is fixed at the terminal time t_1, then a complete set of costate final condition is available. Under these conditions this problem has a closed form solution. Let the final condition on the costate be $P(t_1)=\pi_f$ Then the costate is $P(t)=e^{-A'(t-t_1)}\pi_f$ and the optimal control is

$$u^*(t)=-\text{dez}[q'(t)e^{AT}\pi_f]=-\text{dez}[q'_f(t)\pi_f], \tag{28}$$

and the state is

$$x(t)=e^{At}[\xi-\int_0^t q(t)\text{dez}[q'(t)e^{A'T}\pi_f]]dt. \tag{29}$$

4. Condition for Convergence.

The sufficient conditions for convergence of this method and the proof are introduced by an ealier paper[5]. The purpose of this section is to apply the sufficient condition to the approximate operator $T_k(\pi_j)$ of Eq. (20) and to write out the required expression.

The first derivatives of $T_k(\pi_j)$ is given as Eq. (23). Here the task is to evaluate or bound certain norms.

Let

$$\Gamma_{k+1}\equiv[T_{k+1}^{(1)}(\pi_j)]^{-1}, \tag{30}$$

then

A norm$=||\Gamma_{k+1}\ T_{k+1}(\pi_j)||$

B norm$=||\Gamma_{k+1}+T_{k+2}^{(2)}(\pi_j)||$

are the required norms. B norm is to be evaluated over all possible vector π_f belonging to the n dimenional vector space.

Using the definition of operator T_k, its derivatives lead to the expanded formulas

$$\Gamma_{k+1}=-[-[e_{jj}:e_{jl}][\alpha_{k+1}W_f(T)+\int_0^T q(t)q'_f(t)u_{k+1}^{(1)}[q'_f(t)\pi_j+q'_l(t)\pi_l]dt]]^{-1}. \tag{31}$$

There remains the problem of searching over the space of cost initial condition vectors for the one which yields the largest value of the norm. Actually it is only necessary to search in a sphere around π_f, but the radius of this sphere is not known before hand,

If the argument of (30) is a vector S_i,

A norn$=||S_i||$

one can use

$$\text{A norm}=\max_i |S_i|. \tag{32}$$

For B norm, a third order tensor must be handled. In this case, the argument of the norm is of the form S_{ijk}

$$\text{B norm} = \|S_{ijk}\|,$$

B norm can be bounded by the expression

$$\text{B norm} \leq \max_i \sum_i \sum_j |S_{ijk}|. \tag{33}$$

Kantorvich's theorem[10] then guarantees convergence of the sequence method if

$$h \equiv \text{A norm} \cdot \text{B norm} < \frac{1}{2} \tag{34}$$

Then it is necessary that this information will be used in design the sequence of approximate operators.

5. Discussion on its Physical Characteristic.

The prime objectives of this section is to clarify the physical significance of the problem of this type and to envisage the relationship previously developed. Regarding this kind of problem the type of the following equations can be formulated.

a. Let's consider the state equation as follows;

$$x = f(x, u) \tag{35}$$

b. A fixed time interval

$$t \epsilon [0, T] \tag{36}$$

c. Initial boundary condition

$$x(0) = \xi \tag{37}$$

d. The control variable must statisfy a constraint

$$|u(t)| \leq C \text{ for all } t \epsilon [0, T] \tag{38}$$

e. The cost function is

$$J(u) = \int_0^T f(x, u) dt. \tag{39}$$

Then, it is desired to find a control variable $u^*(t)$ that satisfies the constraint and minimizes the cost function.

Using the penalty function technique as a treatment of constraints[5; 15, 16, 17] which does not depend on the final state vector $X_i(T)$, yield $\pi_i \equiv ◎$(◎: zero element). If none of the state variables are fixed at the terminal time T, then a complete set of costate final condition is variable. Under these conditions, the problem has a closed form solution. Let the final condition on the costate be

$$P(T) = \pi_f. \tag{40}$$

Now we consider how the state end point will be determined from the geometrical point of view. In the $n+1$ dimensional space hyperplane group with parameter d is given by

$$\sum_{i=0}^{n} b_i x_i(T) = d_j, (i = 1, 2 \cdots {}^{j} = 1, 2 \cdots) \tag{41}$$

As we can see in Fig. 1, the hyperplane group is paralleded each other and meets at right angles at the position $d_j/\|b\|$ in the direction $(b_0, b_1, b_2, \cdots\cdots b_n)$ from the zero point.

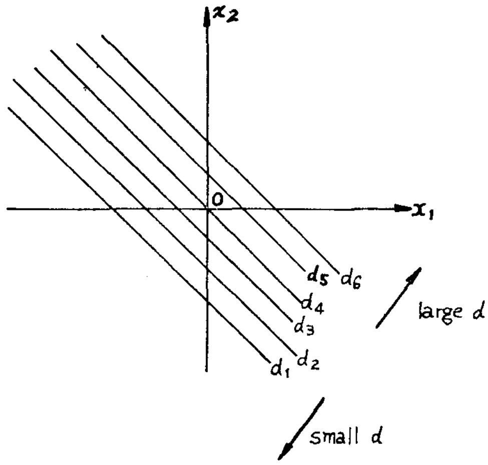

Fig. 1. Hyperplane group

When the allowable control is operated on a process at the initial condition ξ, t seconds, the state points shape a kind of set, that is, maximum isochronal surface. This is a closed curved surface with parameter t in $n+1$ dimensional space and given by

$$C(x) = t. \tag{42}$$

On the maximum isochronal surface the allowable control that can reach a desirable point is found to be only one. Now let it be the equal of T, the maximum isochronal surface $C(x) = T$ is decided as may be observed from Fig. 2. There are two that come in contact with the maximum isochronal surface.

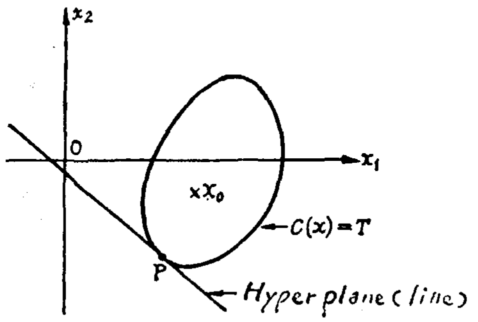

Fig. 2. Terminal end point

If taking the smaller side of d, P is choosed as the contacted point, this P is the terminal state of $x(T)$.

The optimal trajactory is the locus through P from the initial condition in the contacted point of two maximum isochronal surface and the optimal control of this problem is to operate the state point not to derail on the

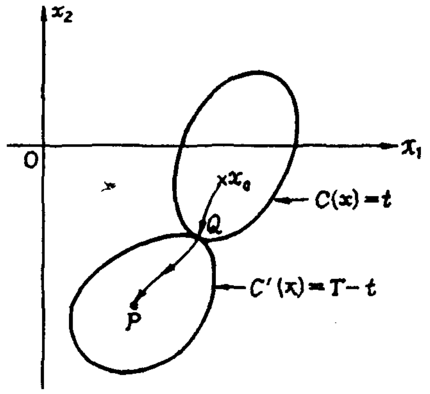

Fig. 3. State point in free time

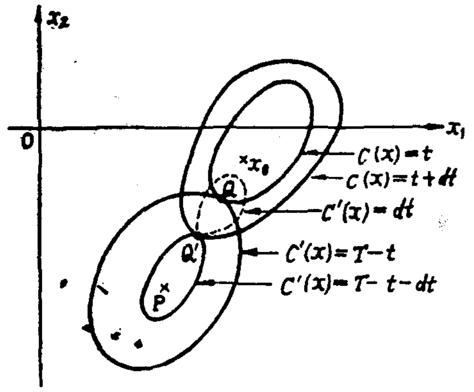

Fig. 4. Transient of state point

optimal trajactory as in Fig. 3. Fig. 4 shows both the maximum isochronal surface and the contacted point in time t. In time $t+dt$ two maximum isochronal surfaces are given by

$$C(x)=t+dt$$
$$C'(x)=T-t-dt, \tag{43}$$

and then another contacted point Q' comes about on the curved surface $C'(x)$ and the state point comes to move from Q to Q' within dt. According to a free control besides the domain of the allowable limitation, the state point exists in the inside of $C(x)=t+dt$, such control $u^*(t)$ is to be the optimal control in time t. Then the following relations are obtained:

$$\frac{\partial C}{\partial x_0}f_0+\frac{\partial x}{\partial x_1}f_1+\cdots\cdots+\frac{\partial C}{\partial x_n}f_n=1 \tag{44}$$

Now time variant of $\frac{\partial C}{\partial x_1}$ happend with the transient of the state point through the optimal trajactory is

$$\frac{d}{dt}\left(\frac{\partial C}{\partial x}\right)=\frac{\partial^2 C}{\partial x_1\partial x_0}f_0+\cdots+\frac{\partial^2 C}{\partial x_1\partial x_n}f_n \tag{45}$$

Besides, partial derivatives of Eq. (44) with x_1 is

$$\frac{\partial^2 C}{\partial x_0\partial x_1}f_0+\frac{\partial^2 C}{\partial x_1\partial x_1}f_1+\cdots\cdots+\frac{\partial^2 C}{\partial x_n\partial x_1}f_n$$
$$=-\frac{\partial C}{\partial x_0}\frac{\partial f_0}{\partial x_1}\cdots\cdots-\frac{\partial C}{\partial x_n}\frac{\partial f_n}{\partial x_1} \tag{46}$$

generally, the following form yields

$$\frac{d}{dt}\left(\frac{\partial C}{\partial x_i}\right)=-\sum_{j=0}^{n}\frac{\partial C}{\partial x_j}\frac{\partial f_j}{\partial x_i},\ (i=0,1,2,\cdots,n) \tag{47}$$

According to maximum principle

$$\mathrm{P}=-\sum_{j=0}^{n}P_j\left(\frac{\partial f_j}{\partial x_i}\right),\ (i=0,1,2,\cdots\cdots,n). \tag{48}$$

Then, the following relation from above relations yields

$$\frac{\partial C}{\partial x_0}f_0+\frac{\partial C}{\partial x_1}f_1+\cdots\cdots+\frac{\partial C}{\partial x_n}f_n$$
$$\equiv \mathrm{H}[x,u,p,t,]\leqq 1. \tag{49}$$

which denotes that the optimal control is to make Hamiltonian H be maximum, this relations were pointed out by ICHIKAWA[14).

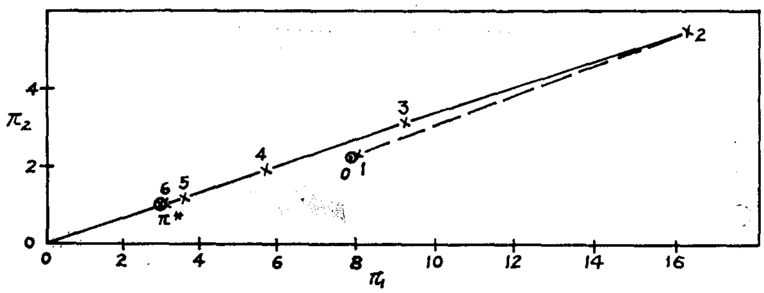

Fig. 5. Graph of the Sequence $\{\pi_k\}$

6. Application

The purpose was to examine the effect of the sequence. Some of the more enlightening computer results in some physical models are enumerated.

Example 1. Double exponential plant. This plant has two real poles, one of them unstable.

$$A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad b=\begin{pmatrix}1\\ 1\end{pmatrix} \quad T=2.0$$

$$\pi^*=\begin{pmatrix}3\\ 1\end{pmatrix} \quad M=100$$

M is the mode of operation.

This costate initial condition was choosen to give a control history of -1, 0, -1. The sequence of vector in Fig. 5 converged nicely to the true value of π^*. Notice that they do not quite lie in a straight line. They seem to first increase on one line uutil $\alpha\to 0$ and then decrease on a line through the origin. So the linear solution is not proportional to π^*.

Example 2. The quadrupole plant.

With a combination of the double exponential plant and the single oscillator plant, a symmetric arrangement of four poles is obtained.

$$A=\begin{pmatrix}1 & 0 & & \\ 0 & -1 & & 0\\ & & 0 & 1\\ 0 & & -1 & 0\end{pmatrix} \quad b=\begin{pmatrix}1\\ -1\\ 0\\ 1\end{pmatrix} \quad \begin{matrix}T=2.0\\ M=100\end{matrix}$$

$$\pi^*=\begin{pmatrix}-1\\ 0\\ 0\\ 3\end{pmatrix}$$

When W=1 in the oscillator portion this plant was called a quadrupole. For other

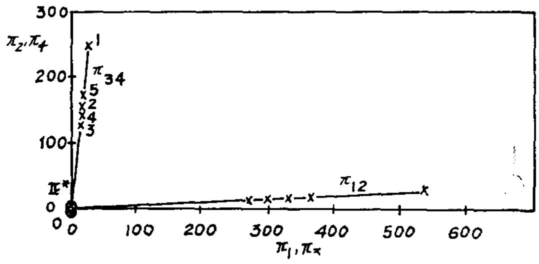

Fig. 6. Graph of the Sequence $\{\pi_k\}$

values of W the plant was called a quadrupole.

The designed control history for this run was -1, 0, $+1$, with the two degrees of freedom this gives, both the π_{12} and the π_{34} plots shown in Fig. 6 moved far out along straight lines. Actually the sequence first moved out, then part way back, and then out again, giving the best example found of the way in which the sequence $\{\pi_k\}$ can depart from the vector π^*. Even with this behavior all the operators converged.

Example 3. The quadrupole oscillator plant.

As a variation, the quadrupole plant is investigated with w=4

$$A=\begin{pmatrix}1 & 0 & & \\ 0 & -1 & & 0\\ & & 0 & 4\\ 0 & & -4 & 0\end{pmatrix} \quad b=\begin{pmatrix}1\\ -1\\ 0\\ 1\end{pmatrix} \quad T=2.0$$

$$\pi^*=\begin{pmatrix}-3\\ 0\\ .5\\ 1.5\end{pmatrix} \quad M=100$$

The largest element of π^* is the first, and in Fig. 7 it is the element π, in the sequence of vectors $\{\pi_k\}$ which show the largest magnitudes.

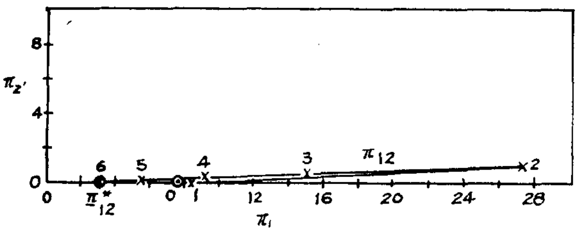

Fig. 7. Graph of the Sequence $\{\pi_k\}$

7. Conclusion.

It is possible to treat the boundary condition with non terminal state by an approach similar to that used in the new computing method[5] for all the terminal condition were fixed. However, one can not handle the relations with some assurance.

There may be more than one external solution, the operator $T(\pi_f)$ may be so complex that sequence method is difficult to handle; an analytic expression may not be available for the form of the optimal control $u^*(t)$; and so on. One conclusion is that the strength of the method is its flexibility and the problem can be solved under such an assumption as the corresponding costate vector $P^*(t)$ exists.

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