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A New Acceleration Method of Additive Angular Dependent Rebalance with Extrapolation for Discrete Ordinates Transport Equation

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Abstract

A new extrapolation method is developed and applied to the additive angular dependent rebalance (AADR) acceleration for discrete ordinates neutron transport calculations. With this extrapolation, the convergence of AADR solution for distinct discretizations between the high-order and low-order equations is remarkably improved and thus the "inconsistent discretization problem" is resolved. Fourier analysis is also performed to find the optimal extrapolation and weighting parameters, which give the smallest spectral radius. The numerical tests demonstrate that the AADR with extrapolation works well as predicted by the Fourier analysis.

Key Words : neutron, transport, calculation, acceleration, extrapolation, fourier analysis

1. Introduction

It is known that most linear acceleration methods based on low-order equation for correction, for convergence, require the low-order preconditioning equation to be discretized consistently with the discretizaion of the high-order transport sweep [1]. On the other hand, nonlinear methods do not, for rapid convergence, require the low-order equation to be discretized consistently with the high-order equation. To avoid the above consistency requirement, an extrapolation concept is considered in this paper for the additive angular dependent rebalance (AADR) method [2][3][4] which is a linear form of ADR [5][6]. An extrapolation concept was used previously in the diffusion synthetic acceleration (DSA) method [7]. But, a different extrapolation is considered in this study. The new extrapolation applied to AADR affects the convergence of the solution drastically and it provides stability with better performance of AADR with inconsistent discretization. Fourier analyses as well as numerical tests for various cases are performed and the optimal parameters, which give the smallest spectral radius, are also found from Fourier analysis. This work was presented in preliminary form in Ref. 8.

2. Description of AADR with Extrapolation

Basic equations of the additive angular dependent rebalance (AADR) with S2-like rebalance are given as follows. The high-order equation, which provides angular flux ($\psi^{l+1/2}$), is given as

$$\mu \frac{d\psi^{l+1/2}}{dx} + \sigma\ \psi^{l+1/2} = \sigma_s \phi^l + q(x), \tag{1}$$

where l is an iteration index, μ is a directional angular cosine, σ is a macroscopic total cross section, σ_s is a macroscopic scattering cross section, and $q(x)$ is an external source. The scalar flux ($\phi^{l+½}$) is obtained by integrating angular flux over angular domain:

$$\phi^{l+1/2} = \frac{1}{2}\int_{-1}^{1} \psi^{l+1/2} d\mu. \tag{2}$$

To derive the low-order equation for acceleration, changing all indices in Eq. (1) into $l+1$,

$$\mu \frac{d\psi^{l+1}}{dx} + \sigma\ \psi^{l+1} = \sigma_s \phi^{l+1} + q(x), \tag{3}$$

and then subtracting Eq. (1) from Eq. (3), we obtain

$$\mu \frac{d(\psi^{l+1} - \psi^{l+1/2})}{dx} + \sigma\ (\psi^{l+1} - \psi^{l+1/2}) = \sigma_s(\phi^{l+1} - \phi^l). \tag{4}$$

We then integrate the resulting equation over half-angular space with a weighting function ($W(\mu)$) to obtain

$$k\ \frac{df_+^{l+1}}{dx} + \sigma\ f_+^{l+1} = \sigma_s(\phi^{l+1} - \phi^l), \quad \mu > 0, \tag{5}$$

$$-k\ \frac{df_-^{l+1}}{dx} + \sigma\ f_-^{l+1} = \sigma_s(\phi^{l+1} - \phi^l), \quad \mu < 0, \tag{6}$$

where k is a weighting parameter which is defined as

$$k\ = \int_0^1 \mu W(\mu) d\mu \Big/ \int_0^1 W(\mu) d\mu, \tag{7}$$

and rebalance factors are defined as

$$\begin{aligned} f_+^{l+1} &= \psi^{l+1} - \psi^{l+1/2}, \quad \mu > 0, \\ f_-^{l+1} &= \psi^{l+1} - \psi^{l+1/2}, \quad \mu < 0. \end{aligned} \tag{8}$$

The optimal weighting parameter (k) can be found from Fourier analysis. Finally, the scalar flux is updated with the previous scalar flux which is the result of the high-order equation and rebalance factors which are the solutions of the low-order equation:

$$\phi^{l+1} = \phi^{l+1/2} + \frac{f_+^{l+1} + f_-^{l+1}}{2}. \tag{9}$$

An extrapolation concept for AADR is first considered with scalar flux ϕ^{l+1}. Thus, we may modify Eq.(9) as

$$\phi^{l+1} = \alpha(\phi^{l+1/2} + \frac{f_+^{l+1} + f_-^{l+1}}{2}) + (1-\alpha)\phi^l, \tag{10}$$

where α is an extrapolation parameter. A similar idea was used before to get better performance of the diffusion synthetic acceleration (DSA).[7] But when applied to AADR, it turned out that the method does not work, because the spectral radius can be larger than unity. So a new extrapolation concept with $\phi^{l+1/2}$ (not with ϕ^{l+1}) is devised in this paper. Thus, Eq.(2) is replaced by

$$\phi^{l+1/2} = \alpha \frac{1}{2}\int_{-1}^{1} \psi^{l+1/2} d\mu + (1-\alpha)\phi^l. \tag{11}$$

The way α is introduced in Eq. (11) avoids multiplication of the extrapolation parameter with the rebalance factors. This extrapolation concept is found to provide better results from Fourier analysis and numerical tests.

3. Continuous Fourier Analysis of AADR with Extrapolation

Fourier analysis is performed to investigate the efficiency of the extrapolation. Let us define Fourier ansatz as

$$\begin{aligned}
\psi^{l+1/2} &= A\,\omega^l Exp(j\lambda x\ ),\\
\phi^l &= B\omega^l Exp(j\lambda x\ ),\\
\phi^{l+1/2} &= D\omega^l Exp(j\lambda x\ ),\\
f_{\pm}^{l+1} &= F_{\pm}\omega^l Exp(j\lambda x\ ),
\end{aligned} \tag{12}$$

where j = $\sqrt{-1}$ And several assumptions, without loss of generality, are given as

$$\sigma = 1, \quad \sigma_s = c, \quad q(x) = 0. \tag{13}$$

Then, Eq. (1) becomes

$$(j\lambda\mu + 1)A = cB, \tag{14}$$

and Eq. (11)

$$D = \frac{\alpha}{2}\int_{-1}^{1}\frac{cB}{j\lambda\mu + 1}d\mu + (1-\alpha)B. \tag{15}$$

Low-order equations (5) and (6) are also expressed as

$$(j\lambda k + 1)F_{+} = c(\omega - 1)B, \tag{16}$$

$$(-j\lambda k + 1)F_{-} = c(\omega - 1)B. \tag{17}$$

Using the above two Eqs. (16) and (17), we obtain

$$F_{+} + F_{-} = \frac{(\omega-1)c}{1+\lambda^2k^2}B. \tag{18}$$

Finally, using Eqs. (14) and (18), Eq. (9) becomes

$$\omega B = \frac{\alpha}{2}\int_{-1}^{1}\frac{cB}{j\lambda\mu+1}d\mu + \frac{(\omega-1)c}{1+\lambda^2k^2}B + (1-\alpha)B, \tag{19}$$

and arranging for eigenvalue (ω), then we obtain

$$\omega\left(1 - \frac{c}{1+\lambda^2k^2}\right) = \frac{\alpha}{2}\int_{-1}^{1}\frac{c}{1+\lambda^2\mu^2}d\mu - \frac{c}{1+\lambda^2k^2} + (1-\alpha). \tag{20}$$

Multiplying the denominator on both sides of Eq. (20), Eq. (20) can be expressed as

$$\omega(1+\lambda^2k^2-c) = \frac{1}{2}\int_{-1}^{1}\frac{c(\alpha\lambda^2k^2+\alpha-1-\lambda^2\mu^2)+(1-\alpha)(1+\lambda^2k^2)(1+\lambda^2\mu^2)}{1+\lambda^2\mu^2}d\mu \tag{21}$$

Finally, the eigenvalue is expressed in the following inequality as

$$\omega \le \frac{1}{2}\int_{-1}^{1}\frac{1+\dfrac{(1-\alpha)\lambda^2k^2-\alpha}{\lambda^2k^2}\lambda^2\mu^2}{1+\lambda^2\mu^2}d\mu, \tag{22}$$

and the spectral radius (ω), maximum of the eigenvalues, is obtained in an analytic form:

$$\rho = \sup_\lambda\left|\frac{1}{2}\int_{-1}^{1}\frac{1+A(\alpha,\lambda)\lambda^2\mu^2}{1+\lambda^2\mu^2}d\mu\right| = \sup_\lambda\left|A(\alpha,\lambda) + \frac{(1-A(\alpha,\lambda))\arctan(\lambda)}{\lambda}\right|, \tag{23}$$

where $A(\alpha,\lambda)=((1-\alpha)\lambda^2k^2-\alpha)/(\lambda^2k^2)$.
Eq. (23) can be rewritten as

$$\rho = \sup_\lambda\left|\alpha\left(A(1,\lambda) + \frac{(1-A(1,\lambda))\arctan(\lambda)}{\lambda}\right) + (1-\alpha)\right|. \tag{24}$$

The S_4-like rebalanced AADR is also analyzed by Fourier analysis. The lower-order equations of S_4-like rebalanced AADR are given as

$$l_m\frac{df_{m+}^{l+1}}{dx} + \sigma\, f_{m+}^{l+1} = \sigma_s(\phi^{l+1} - \phi^l), \quad \mu > 0, \tag{25}$$

$$-l_m\frac{df_{m-}^{l+1}}{dx} + \sigma\, f_{m-}^{l+1} = \sigma_s(\phi^{l+1} - \phi^l), \quad \mu < 0, \quad m = 0,1, \tag{26}$$

where

$$\begin{aligned}
l_0 &= \int_0^{\delta}\mu W(\mu)d\mu \Big/ \int_0^{\delta}W(\mu)d\mu,\\
l_1 &= \int_{\delta}^{1}\mu W(\mu)d\mu \Big/ \int_{\delta}^{1}W(\mu)d\mu, \quad 0<\delta<1.
\end{aligned} \tag{27}$$

The scalar flux is updated as

$$\phi^{l+1} = \phi^{l+1/2} + \frac{\delta(f_{0+}^{l+1}+f_{0-}^{l+1}) + (1-\delta)(f_{1+}^{l+1}+f_{1-}^{l+1})}{2}. \tag{28}$$

For simplicity, the constant δ is chosen as $1/\sqrt{2}$ not δ in this study. The spectral radius with extrapolation is given in a general form such as

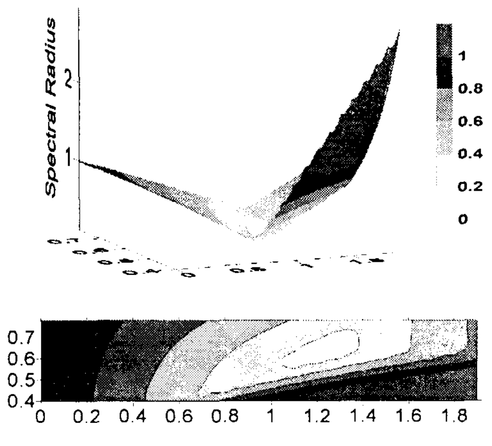

Fig. 1. Optimal Weighting and Extrapolation Parameters in S_2-like AADR

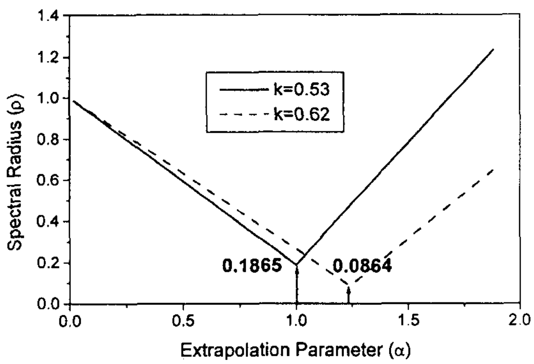

Fig. 2. Spectral Radius for Various Extrapolation Parameters in S_2-like AADR

$$\rho = \sup_{\lambda} c\left|\alpha\left(\frac{\lambda a(\lambda) - \arctan(\lambda)}{(a(\lambda)-1)\lambda}\right) + (1-\alpha)\right|, \qquad (29)$$

where $a(\lambda)=1/(1+\lambda^2k^2)$ for S_2-like AADR and $a(\lambda)=(1/(1+\lambda^2 l^2_0)+1/(1+\lambda^2 l^2_1))/2$ for S_4-like AADR.

From the continuous Fourier analysis for various rebalance methods, we can find optimal α, which provides the smallest spectral radius:

i) spectral radius (S_2-like rebalanced AADR): 0.1865 (α=1.0),

ii) spectral radius (S_2-like rebalanced AADR): 0.0864 (α=1.2),

iii) spectral radius (S_4-like rebalanced AADR): 0.0485 (α=1.0),

iv) spectral radius (S_4-like rebalanced AADR): 0.0141 (α=1.1).

Fig. 1 shows spectral radii for various extrapolation parameters and weighting parameters from continuous Fourier analysis. Fig. 2 depicts spectral radii of AADR with extrapolation for various weighting parameters. We can find optimal and k in this figure, which provides smallest spectral radius, 0.0864, when approaches about 1.2.

4. Inconsistent Discretization of AADR

The convergence of linear acceleration methods may be poor, if different spatial schemes are used for the high-order and low-order equations. In this study, it is shown by the Fourier analysis and numerical tests that AADR with extrapolation can mitigate this inconsistent discretization problem. In other words, in contrast to DSA, AADR does not have difficulty in discretizing the low-order equation consistently, because the low-order equation of AADR has the same form as the high-order equation. Furthermore, AADR with extrapolation allows inconsistent discretization. We have chosen four cases of S_2-like rebalanced AADR with step difference (SD) scheme and diamond difference (DD) scheme. If the consistent discretization with diamond difference (DD) scheme is considered on both the high-order and the low-order equations, the high-order equations are derived as :

$$\mu_n \frac{\psi^{l+1/2}_{ni+1/2} - \psi^{l+1/2}_{ni-1/2}}{\Delta_i} + \sigma_i \frac{\psi^{l+1/2}_{ni+1/2} + \psi^{l+1/2}_{ni-1/2}}{2} = \sigma_{si}\phi^l_i + q_i, \quad (30)$$

$$\phi^{l+1/2}_i = \frac{1}{4}\sum_{n=1}^{N} w_n(\psi^{l+1/2}_{ni+1/2} + \psi^{l+1/2}_{ni-1/2}), \qquad (31)$$

and the low-order equations of S_2-like AADR with

DD scheme are derived as :

$$k\,\frac{f_{+,i+1/2}^{l+1}-f_{+,i-1/2}^{l+1}}{\Delta_i}+\sigma_i\frac{f_{+,i+1/2}^{l+1}+f_{+,i-1/2}^{l+1}}{2}=\sigma_{si}(\phi_i^{l+1}-\phi_i^{l}),\quad \mu_n>0, \tag{32}$$

$$-k\,\frac{f_{-,i+1/2}^{l+1}-f_{-,i-1/2}^{l+1}}{\Delta_i}+\sigma_i\frac{f_{-,i+1/2}^{l+1}+f_{-,i-1/2}^{l+1}}{2}=\sigma_{si}(\phi_i^{l+1}-\phi_i^{l}),\quad \mu_n<0, \tag{33}$$

$$\phi_i^{l+1}=\phi_i^{l+1/2}+\frac{f_{+,i-1/2}^{l+1}+f_{+,i+1/2}^{l+1}+f_{-,i-1/2}^{l+1}+f_{-,i+1/2}^{l+1}}{4}. \tag{34}$$

Inconsistent discretization with diamond difference (DD) scheme for the high-order equation and step difference (SD) scheme for the low-order equation of S_2-like AADR provides the following the low-order equation as :

$$k\,\frac{f_{+,i+1/2}^{l+1}-f_{+,i-1/2}^{l+1}}{\Delta_i}+\sigma_i f_{+,i+1/2}^{l+1}=\sigma_{si}(\phi_i^{l+1}-\phi_i^{l}),\quad \mu_n>0, \tag{35}$$

$$-k\,\frac{f_{-,i+1/2}^{l+1}-f_{-,i-1/2}^{l+1}}{\Delta_i}+\sigma_i f_{-,i-1/2}^{l+1}=\sigma_{si}(\phi_i^{l+1}-\phi_i^{l}),\quad \mu_n<0, \tag{36}$$

$$\phi_i^{l+1}=\phi_i^{l+1/2}+\frac{f_{+,i+1/2}^{l+1}+f_{-,i-1/2}^{l+1}}{2}. \tag{37}$$

The spectral radii obtained from discrete Fourier analysis can be derived. The spectral radii of AADR with various combinations of DD and SD are given as:

a) AADR0 (DD-DD):

$$\rho=\sup_\lambda\left|c\alpha\sum_{n=1}^{N/2}\frac{w_n[(-\chi_n^2+\kappa^2)\cos^2(\tau)]}{\chi_n^2\kappa^2\sin^2(\tau)+\kappa^2\cos^2(\tau)}+(1-\alpha)\right|, \tag{38}$$

b) AADR1 (DD-SD):

$$\rho=\sup_\lambda\left|c\alpha\sum_{n=1}^{N/2}\frac{w_n[-(\kappa+1)\chi_n^2\sin^2(\tau)+\{(\kappa+1)\kappa-\chi_n^2\}\cos^2(\tau)]}{\chi_n^2(\kappa+1)\kappa\sin^2(\tau)+(\kappa+1)\kappa\cos^2(\tau)}+(1-\alpha)\right|, \tag{39}$$

c) AADR2 (SD-DD):

$$\rho=\sup_\lambda\left|c\alpha\sum_{n=1}^{N/2}\frac{w_n[\chi_n\kappa^2\sin^2(\tau)+\{-(\chi_n+1)\chi_n+\kappa^2\}\cos^2(\tau)]}{(\chi_n+1)^2\kappa^2\sin^2(\tau)+\kappa^2\cos^2(\tau)}+(1-\alpha)\right|, \tag{40}$$

d) AADR3 (SD-SD):

$$\rho=\sup_\lambda\left|c\alpha\sum_{n=1}^{N/2}\frac{w_n(\kappa+1)(\chi_n+1)(\kappa-\chi_n)\sin^2(\tau)+(\kappa-\chi_n)(\kappa+\chi_n+1)\cos^2(\tau)}{(\chi_n+1)^2(\kappa+1)\kappa\sin^2(\tau)+(\kappa+1)\kappa\cos^2(\tau)}+(1-\alpha)\right|, \tag{41}$$

where $\tau=\lambda\Delta/2$, $\kappa=2k/\Delta$, $\chi_n=2\mu_n/\Delta$. Here DD-SD means that the diamond difference scheme is applied for high-order equation and the step difference scheme for low-order equation.

Discrete Fourier analysis is also performed for the S_4-like rebalanced AADR with diamond difference (DD) scheme for the high-order equation and step difference (SD) scheme for the low-order equation. The spectral radius is given as

$$\rho=\sup_\lambda\left|\alpha c\sum_{n=1}^{N/2}w_n\frac{T}{S}+(1-\alpha)\right|, \tag{42}$$

where

$$S=A_1\cos^4(\tau)+B_1\sin^2(\tau)\cos^2(\tau)+C_1\sin^4(\tau),$$
$$T=A_2\cos^4(\tau)+B_2\sin^2(\tau)\cos^2(\tau)+C_2\sin^4(\tau),$$
$$A_1=L_0^2+L_1^2-L_0-L_1,\quad B_1=2L_0^2L_1^2-L_0L_1(L_0+L_1)+M^2A_1,$$
$$C_1=M^2(2L_0^2L_1^2-L_0L_1(L_0+L_1)),$$
$$A_2=A_1-2M^2,\ B_2=2L_0^2L_1^2-L_0L_1(L_0+L_1)-M^2(L_0^2+L_1^2+L_0+L_1),$$
$$C_2=-M^2L_0^2L_1^2(L_0+L_1),$$
$$L_0=(2l_0/\Delta_i+1),\quad L_1=(2l_1/\Delta_i+1),\quad M=(2\mu_n/\Delta_i).$$

Spectral radii for various cases of S_2-like rebalanced AADR without extrapolation are depicted in Fig. 3.

The spectral radii of consistently discretized AADR (DD-DD, SD-SD) are very small for various mesh sizes. But in the case of inconsistent discretization (DD-SD, SD-DD), the spectral radii approach around unity for large mesh sizes, which will take a large number of iterations or may not

Table 1. Number of Iterations and Computing Times for S2-like AADR

	AADR (DD[a]-DD[b])	AADR (DD-SD)	AADR (SD-DD)	AADR (SD-SD)
Source Iteration	39171[c] 0.9997[d] (19.2 sec)[e]	39171 0.9997 (19.2 sec)	6549 0.9986 (2.67 sec)	6549 0.9986 (2.67 sec)
AADR without Extrapolation	10 0.0908 (0.03 sec) α=1.00 k=0.55 (W=|μ|+1.17)	566 0.9701 (1.42 sec) α=1.00 k=1.86 (W=|μ|-0.44)	98 0.8422 (0.13 sec) α=1.00 k=1.22 (W=|μ|-0.38)	7 0.0348 (0.03 sec) α=1.00 k=0.51 (W=|μ|+7.83)
AADR with Extrapolation	8 0.0245 (0.02 sec) α=1.24 k=0.63 (W=|μ|+0.14)	6 0.0264 (0.02 sec) α=-9.00 k=-4.30 (W=|μ|-0.52)	5 0.0105 (0.02 sec) α=11.4 k=5.70 (W=|μ|-0.48)	6 0.0111 (0.03 sec) α=1.24 k=0.63 (W=|μ|+0.14)

[a]: Solver for high-order equation, [b]: Solver for low-order equation, [c]: Number of iterations,
[d]: *Numerical spectral radius*, [e]: *Calculation on SUN-ULTRA1 system*

converge. With optimal extrapolation parameter (α) and optimal weighting parameter (k), the spectral radii of S_2-like AADR (DD-SD) are depicted in Fig. 4. Note that the spectral radii are very small for large mesh sizes if optimal parameters are used in inconsistently discretized AADR.

5. Numerical Tests and Results

The first test problem is an isotropic homogeneous slab, 100 cm wide with scattering ratio (c) of 1, total cross section of 1 cm^{-1}, and a uniform source of 1 cm^{-3}s^{-1}. The vacuum boundary conditions are imposed on both sides and S_{16} Gauss-

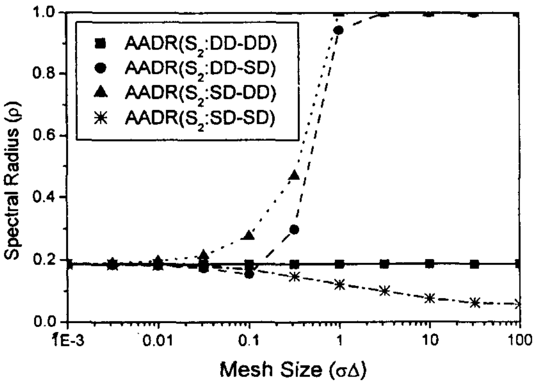

Fig. 3. Spectral Radius for Various Mesh Sizes (k=0.53, α=1.0, c=1.0)

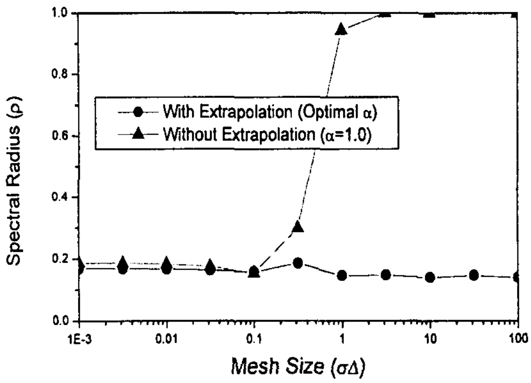

Fig. 4. Spectral Radius for Inconsistently Discretized S_2-like AADR (DD-SD)

Table 2. Cross Section for Heterogeneous Problem

Cross Section	Region 1	Region 2	Region 3	Region 4	Region 5
σ (cm^{-1})	50	5	0.001	1	1
σ_s (cm^{-1})	50	5	0.001	0.9	0.9
Q (#/cm^{3}s)	50	0	0	1	0.0

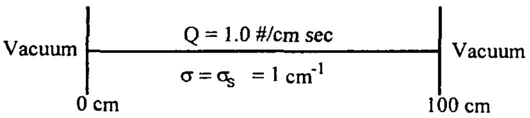

Fig. 5. Configuration of Homogeneous Test Problem

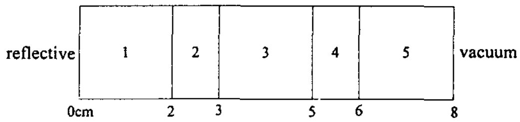

Fig. 6. Configuration of Heterogeneous Problem

Legendre quadrature is used. The mesh size is chosen as 10 cm and convergence criterion is given as 1.0E-9. Fig. 5 shows the configuration of homogeneous test problem. Table 1 shows the number of iterations and computing times for various cases of S_2-like rebalanced AADR. AADR with extrapolation shows better results than AADR without extrapolation. For example, S_2-like rebalanced AADR with inconsistent discretization (DD-SD) requires 566 iterations without extrapolation but 6 iterations with extrapolation. The Fourier analysis for this case indicates that its spectral radius without extrapolation is nearly unity, but if extrapolation is considered, its spectral radius is less than 0.13953 as shown in Fig. 4. We also obtain similar results with S_4-like rebalanced AADR. All optimal parameters are found from Fourier analysis and the numerical results are in good agreement with those of discrete Fourier analysis.

The second test problem is a heterogeneous problem depicted in Fig. 6 and Table 2 shows the cross sections. The convergence criterion is 1.0E-9 and S_{16} Gauss-Legendre quadrature is used. This problem is a modification of the Reed' s test problem[9] by changing the scattering cross sections in regions 1 and 2. In fact, any acceleration methods (DSA, AADR, etc) would not provide any gains for the Reed' s problem, because it is a nearly-pure-absorption problem. Thus, the scattering cross sections of regions 1 and 2 are increased in this problem to test the effectiveness of acceleration methods. Table 3 shows the results of calculation with S_2-like and S_4-like AADRs which also provide good results even if inconsistent discretization is used The diffusion synthetic acceleration (DSA) method in the DANTSYS[10] code system is compared in this test. In principle, to get the better performance with inconsistent discretization, different weighting parameters should be used in each region. In this test problem, region 1 is optically much thicker than other regions, thus only the weighting parameters of region 1 are important. The weighting parameters used for this region were determined from the results of discrete Fourier analysis. In the other regions which are comparatively optically thinner, the weighting parameters obtained for the case of consistent discretization without extrapolation were used. As the problem becomes optically thick, the

Table 3. Number of Iterations and Computing Times for AADR

mesh size		S_2-like AADR		S_4-like AADR	
		DD[a]-DD[b]	DD-SD	DD-DD	DD-SD
0.1cm	α=1.00	12[c] (0.10 sec)[d] k=0.56	327 (2.32 sec) k=0.56	10 (0.10 sec) l_0=0.35 l_1=0.80	796 (19.1 sec) l_0=3.0 l_1=3.0
	Extrapolation in Region 1	11 (0.12 sec) k=0.56 α=1.05	18 (1.3 sec) k=−1.76 α=−3.5	8 (0.15 sec) l_0=0.35 l_1=0.80 α=1.05	20 (0.73 sec) l_0=−2.1 l_1=−0.7 α=−2.7
c.f.		Source Iteration : 155534 (355 sec) DSA[e] : 12 (0.20 sec)			
0.01cm	α=1.00	12 (1.13 sec) k=0.56	17 (4.44 sec) k=0.56	10 (1.34 sec) l_0=0.35 l_1=0.80	21 (2.48 sec) l_0=0.36 l_1=0.80
	Extrapol-ation in Region 1	11 (0.95 sec) k=0.56 α=1.05	12 (2.60 sec) k=0.41 α=0.8	8 (2.77 sec) l_0=0.35 l_1=0.80 α=1.05	11 (1.53 sec) l_0=0.23 l_1=0.51 α=0.7
	c.f.	Source Iteration : 155541 (3411 sec) DSA : 12 (1.70 sec)			

[a]: Solver for high-order equation, [b]: Solver for low-order equation, [c]: Number of iterations,
[d]: Calculation on SUN-ULTRA1 system, [e]: DANTSYS code system (solver module).

convergence will be poor without extrapolation as shown in Fig. 4. Therefore, more accurate weighting parameters with extrapolation were used only in optically thick region.

The solutions of inconsistently discretized AADRs converge to those of high-order solvers, but in the case of nonlinear acceleration methods, the solutions approach those of low-order solvers. Thus, when AADR with extrapolation is applied to realistic problems, we should choose a highly accurate scheme as the solver of the high-order equation and a simple scheme such as step difference scheme as the solver of the low-order equation. In this study, the preconditioned bi-conjugate gradient stabilized (PBi-CGSTAB) algorithm with the "transport sweep product (TSP)" preconditioner [3][4] was used to solve the low-order equation. A nice property of the TSP preconditioner is that it is already in LU decomposed form so that the low-order solution can be obtained directly. Thus, the low-order equation is solved efficiently and even a negative value of the weighting parameter k does not cause any numerical problem.

6. Conclusions

A new extrapolation concept is applied to the AADR acceleration method, resulting in remarkable improvement in convergence and the inconsistent discretization problem resolved.
Continuous and discrete Fourier analyses show that AADR with extrapolation provides significantly improved convergence. Even with inconsistent discretizations, the AADR with extrapolation works well and provides fast convergence. Optimal parameters (α, k) can be obtained from Fourier analysis and they are demonstrated by numerical results. Since it is usually known that the extrapolation concept is more effective in two- and three-dimensional problems, where α could be estimated during iteration, future work should address online α-adaptation.

As a conclusion, AADR (a linear acceleration method) with the new extrapolation concept does not require the low-order equation to be discretized consistently with the discretization of the high-order transport sweep, in contrast to the case of DSA.

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