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Development of the Discrete-Ordinates, Nodal Transport Methods Using the Simplified Even-Parity Neutron Transport Equation

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Abstract

Nodal transport methods are studied for the solution of two dimensional discrete-ordinates, simplified even-parity transport equation(SEP) which is known to be an approximation to the true transport equation. The polynomial expansion nodal method(PEN) and the analytic function expansion nodal method(AFEN) which have been developed for the diffusion theory are used for the solution of the discrete-ordinates form of SEP equation. Our study shows that while the PEN method in diffusion theory can directly be converted without complication, the AFEN method requires a theoretical modification due to the nonhomogeneous property of the transport equation. The numerical results show that the proposed two methods work well with the SEP transport equation with higher accuracies compared with the conventional finite difference method.

Key Words : neutron, transport, diffusion, nodal, discrete-ordinates, even-parity, simplified even-parity, scalar flux, angular flux

1. Introduction

Nodal methods have been extremely successful in the solution of the neutron diffusion equation.[1] To achieve the same success in the transport theory many people have applied the various kinds of nodal methods increasingly to the neutron transport equation, mainly with the first-order Boltzmann transport equation. Recently, the simplified even-parity equation(SEP) has emerged as an approximation to the second-order even-parity equation(EP) which is the other type of the transport equation.[2,3] The SEP gives more robust solution and turned out to be more efficient to computationally solve than the regular EP equation due to the reduction in angular domain by another half, directionally uncoupled reflective boundary conditions, simple diffusion operator in each discrete direction, etc.. Most importantly regarding this study, the elliptic operator of SEP is the same as that of the diffusion equation, so it will be easier to implement the conventional nodal diffusion methods into SEP rather than into the other types of transport equations. In this work, we develope the nodal methods for the solution of SEP using the polynomial expansion nodal method(PEN) and the analytic function expansion nodal method(AFEN). Those two methods are selected as candidates, since they are acknowledged as superior to the other methods and the various limitation caused by the transverse integration can be avoided.[4,5] The goal of this work is to develope the PEN and AFEN transport methods which achieve the same success in SEP as they do in the diffusion equation. In this work we are not greatly concerned about the computational capability of the codes such as accuracy and efficiency, since all the previous effort contributed to expand the code capability of PEN and AFEN can be directly applied to the solution of SEP.

2. Neutron Transport Equations

2.1. Boltzmann Transport Equation

For the solution of the neutron transport equation considerable work has been continuously devoted to achieve higher accuracy, faster convergence, and better stability. We start with the two dimensional, within group, Boltzmann transport equation for the isotropic scattering and source[8,9] :

$$\mu\frac{\partial\psi(x,y,\mu,\eta)}{\partial x}+\eta\frac{\partial\psi(x,y,\mu,\eta)}{\partial y}+\sigma(x,y)\psi(x,y,\mu,\eta) = \sigma_s(x,y)\phi(x,y)+Q(x,y), \quad -1\le\mu,\eta\le 1. \tag{1}$$

In Eq.(1) $\psi(x, y, \mu, \eta)$ is an angular flux at the position (x, y) having the direction (μ, η) and $\phi(x,y)$ is the scalar flux defined as

$$\phi(x,y)=\int\int\psi(x,y,\mu,\eta)\,d\mu d\eta. \tag{2}$$

Here, σ and σ_s are macroscopic total and scattering cross sections, and $Q(x,y)$ is an isotropic neutron source. [If we consider spatial differencing only, the isotropic fission term can be included in $Q(x,y)$.] Equation (1) is a first-order differential equation for the spatial variables x and y. Combined with the discrete-ordinate method(S_N) in which the angular variables are described by the discrete direction m, the equation is solved for the unknown $\psi_m(x,y)$ usually by the finite difference method(FDM) such as the diamond-difference method(DD). The DD is very efficient in computing time, but it is known to be very weak in robustness. It generally has many limitations; producing negative solutions, showing extreme ray effects and the lack of diffusion limits, etc..[8,9]

2.2. Even-Parity Transport Equation

The even-parity transport equation(EP) is represented by

$$-\mu^2\frac{\partial}{\partial x}\frac{1}{\sigma}\frac{\partial\chi}{\partial x}-\eta^2\frac{\partial}{\partial y}\frac{1}{\sigma}\frac{\partial\chi}{\partial y}+\sigma(x,y)\chi(x,y,\mu,\eta) = \sigma_s(x,y)\phi(x,y)+Q(x,y), \quad 0\le\mu,\eta\le 1, \tag{3}$$

where $\chi(x,y,\mu,\eta)$ is the even-parity angular flux.[9] We should note that the preceding equation is the true transport equation since it has been obtained from the Boltzmann transport equation without using any approximation. Due to the even-parity property of χ over the angle, the angular domain of EP becomes one half of that of the Boltzmann transport equation. Also, the EP can reduce the ray effect by using an appropriate piecewise angular quadrature set.[6] However, the EP has many drawbacks such as the computational inefficiency due to the second-order differential operators for each discrete direction and the additional computation caused by directionally coupled reflective boundary conditions, etc..[3]

2.3. Simplified Even-Parity Transport Equation

Recently, a new equation which is called the simplified even-parity equation(SEP) was proposed as an approximation to the true transport equation.[2,3] :

$$-\mu^2\frac{\partial}{\partial x}\frac{1}{\sigma}\frac{\partial\chi}{\partial x}-\eta^2\frac{\partial}{\partial y}\frac{1}{\sigma}\frac{\partial\chi}{\partial y}+\sigma(x,y)\chi(x,y,\mu,\eta)$$
$$=\sigma_s(x,y)\phi(x,y)+Q(x,y),\quad 0\le\mu,\eta\le 1, \tag{4}$$

where $\chi(x,y,\mu,\eta)$ is the SEP angular flux. In analogy to the simplified P_N equations, the multi-dimensional SEP equations can be obtained by simply generalizing one dimensional second-order derivative in EP equation into multi-dimensions. The SEP is not a true transport equation since the approximation $\chi(\mu,\eta)\simeq\chi(\mu,-\eta)$ is used in the derivation. Though the SEP has a weak mathematical foundation, it has many advantages in numerical computation. Unlike EP there is no mixed derivative terms in SEP and the angular domain is reduced by another half of that of EP. Moreover, the recent work showed theoretically and numerically that the SEP has a guaranteed positivity, higher efficiency with the diffusion synthetic acceleration(DSA), and eliminated ray effects. It is also verified that the SEP becomes the simplified P_N equation by simply replacing two dimensional angular quadratures with one dimensional Gauss quadratures.[3,7] Regarding this study, the elliptic nature of SEP makes it relatively easy to convert the conventional diffusion codes to SEP codes. The code capability such as dimensionality, geometry, time-dependency, eigenvalue searches, etc. can be maintained without extreme complications.

The discrete-ordinates form of the SEP equation should be

$$-\mu_m^2\frac{\partial}{\partial x}\frac{1}{\sigma}\frac{\partial\chi_m^{l+1/2}}{\partial x}-\eta_m^2\frac{\partial}{\partial y}\frac{1}{\sigma}\frac{\partial\chi_m^{l+1/2}}{\partial y}$$
$$+\sigma(x,y)\chi_m^{l+1/2}(x,y)=\sigma_s(x,y)\phi^l(x,y)+Q(x,y) \tag{5}$$

for the direction m $(1\le m\le M)$. Here, l is the iteration index. To solve Eq.(5), the scalar flux ϕ^l must be assumed at first and then recalculated at each source iteration by

$$\phi^{l+1}(x,y)=\sum_{m=1}^{M}\omega_m\chi_m^{l+1/2}(x,y) \tag{6}$$

and the iteration will be continued until the scalar flux converges.

3. Polynomial Expansion Nodal Method for the Simplified Even-Parity Equation

Now we consider PEN method for the solution of Eq.(5). We divide a two dimensional domain into a grid of nodes, whose cross sections are taken to be piecewise constant with change of values permitted only at the node interfaces. In the nodal methods the accuracy depends on how many unknowns are defined in a node. Here, we assume five unknowns at a center and at four surfaces as shown in Fig. 1. Higher accuracy can be attained

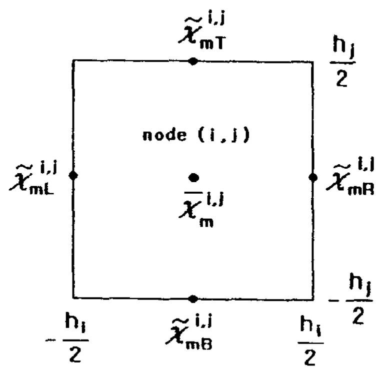

Fig. 1 Rectangular Node (i, j)

by simply increasing the number of unknowns.

We write the SEP equation for the node (i, j) : (iteration index omitted.)

$$-\frac{\mu_m^2}{\sigma^{i,j}}\frac{\partial^2\chi_m^{i,j}}{\partial x^2}-\frac{\eta_m^2}{\sigma^{i,j}}\frac{\partial^2\chi_m^{i,j}}{\partial y^2}+\sigma^{i,j}\chi_m^{i,j}(x,y) = \sigma_s^{i,j}\phi^{i,j}(x,y)+Q^{i,j}, \tag{7}$$

where

$$\phi^{i,j}(x,y)=\sum_{m=1}^{M}\omega_m\chi_m^{i,j}(x,y)\,. \tag{8}$$

Firstly, we expand the intra nodal distribution of the SEP flux in the node (i,j) :

$$\chi_m^{i,j}(x,y)=C_{m1}^{i,j}+C_{m2}^{i,j}x+C_{m3}^{i,j}y + C_{m4}^{i,j}x^2+C_{m5}^{i,j}y^2. \tag{9}$$

Note that the number of terms in the right-hand-side of Eq.(9) is the same as the number of unknowns in the node (i, j). Next, we define a node averaged flux for the direction m as

$$\bar{\chi}_m^{i,j}=\frac{1}{h_ih_j}\int_{-h_j/2}^{h_j/2}\int_{-h_i/2}^{h_i/2}\chi_m^{i,j}(x,y)\,dxdy \tag{10a}$$

and four surface averaged fluxes for the direction m as

$$\tilde{\chi}_{mL}^{i,j}=\frac{1}{h_j}\int_{-h_j/2}^{h_j/2}\chi_m^{i,j}(-h_i/2,y)dy \tag{10b}$$

$$\tilde{\chi}_{mR}^{i,j}=\frac{1}{h_j}\int_{-h_j/2}^{h_j/2}\chi_m^{i,j}(h_i/2,y)dy \tag{10c}$$

$$\tilde{\chi}_{mB}^{i,j}=\frac{1}{h_i}\int_{-h_i/2}^{h_i/2}\chi_m^{i,j}(x,-h_j/2)dx \tag{10d}$$

$$\tilde{\chi}_{mT}^{i,j}=\frac{1}{h_i}\int_{-h_i/2}^{h_i/2}\chi_m^{i,j}(x,h_j/2)dx. \tag{10e}$$

Substituting the expansion function in Eq.(9) to Eqs.(10) and integrating yields

$$\bar{\chi}_m^{i,j}=C_{m1}^{i,j}+\frac{1}{12}h_i^2C_{m4}^{i,j}+\frac{1}{12}h_j^2C_{m5}^{i,j} \tag{11a}$$

$$\tilde{\chi}_{mL}^{i,j}=\frac{1}{h_j}\int_{-h_j/2}^{h_j/2}\chi_m^{i,j}(-h_i/2,y)dy = C_{m1}^{i,j}-\frac{1}{2}h_iC_{m2}^{i,j}+\frac{1}{4}h_i^2C_{m4}^{i,j}+\frac{1}{12}h_j^2C_{m5}^{i,j} \tag{11b}$$

$$\tilde{\chi}_{mR}^{i,j}=\frac{1}{h_j}\int_{-h_j/2}^{h_j/2}\chi_m^{i,j}(h_i/2,y)dy = C_{m1}^{i,j}+\frac{1}{2}h_iC_{m2}^{i,j}+\frac{1}{4}h_i^2C_{m4}^{i,j}+\frac{1}{12}h_j^2C_{m5}^{i,j} \tag{11c}$$

$$\tilde{\chi}_{mB}^{i,j}=\frac{1}{h_i}\int_{-h_i/2}^{h_i/2}\chi_m^{i,j}(x,-h_j/2)dx = C_{m1}^{i,j}-\frac{1}{2}h_jC_{m3}^{i,j}+\frac{1}{12}h_i^2C_{m4}^{i,j}+\frac{1}{4}h_j^2C_{m5}^{i,j} \tag{11d}$$

$$\tilde{\chi}_{mT}^{i,j}=\frac{1}{h_i}\int_{-h_i/2}^{h_i/2}\chi_m^{i,j}(h_i/2,y)dx = C_{m1}^{i,j}+\frac{1}{2}h_jC_{m3}^{i,j}+\frac{1}{12}h_i^2C_{m4}^{i,j}+\frac{1}{4}h_j^2C_{m5}^{i,j} \tag{11e}$$

or

$$\begin{bmatrix}\bar{\chi}_m^{i,j}\\ \tilde{\chi}_{mL}^{i,j}\\ \tilde{\chi}_{mR}^{i,j}\\ \tilde{\chi}_{mB}^{i,j}\\ \tilde{\chi}_{mT}^{i,j}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & h_i^2/12 & h_j^2/12\\ 1 & -h_i/2 & 0 & h_i^2/4 & h_j^2/12\\ 1 & h_i/2 & 0 & h_i^2/4 & h_j^2/12\\ 1 & 0 & -h_j/2 & h_i^2/12 & h_j^2/4\\ 1 & 0 & h_j/2 & h_i^2/12 & h_j^2/4\end{bmatrix}\begin{bmatrix}C_{m1}^{i,j}\\ C_{m2}^{i,j}\\ C_{m3}^{i,j}\\ C_{m4}^{i,j}\\ C_{m5}^{i,j}\end{bmatrix} \tag{12}$$

in the matrix form. The coefficients $C_{mk}^{i,j}\,(k=1,\cdots,5)$ can be obtained using the matrix inversion and the result is

$$\begin{bmatrix}C_{m1}^{i,j}\\ C_{m2}^{i,j}\\ C_{m3}^{i,j}\\ C_{m4}^{i,j}\\ C_{m5}^{i,j}\end{bmatrix}=\begin{bmatrix}2 & -1/4 & -1/4 & -1/4 & -1/4\\ 0 & -1/h_i & 1/h_i & 0 & 0\\ 0 & 0 & 0 & -1/h_j & 1/h_j\\ -6/h_i^2 & 3/h_i^2 & 3/h_i^2 & 0 & 0\\ -6/h_j^2 & 0 & 0 & 3/h_j^2 & 3/h_j^2\end{bmatrix}\begin{bmatrix}\bar{\chi}_m^{i,j}\\ \tilde{\chi}_{mL}^{i,j}\\ \tilde{\chi}_{mR}^{i,j}\\ \tilde{\chi}_{mB}^{i,j}\\ \tilde{\chi}_{mT}^{i,j}\end{bmatrix}. \tag{13}$$

Now, we need to set equations to solve for five unknowns in a node. These equations consist of a node balance equation and four continuity conditions at node interfaces. The node balance equation is obtained by integrating Eq.(7) over the node volume as

$$\frac{1}{h_ih_j}\int_{-h_j/2}^{h_j/2}\int_{-h_i/2}^{h_i/2}\left[-\frac{\mu_m^2}{\sigma^{i,j}}\frac{\partial^2\chi_m^{i,j}}{\partial x^2}-\frac{\eta_m^2}{\sigma^{i,j}}\frac{\partial^2\chi_m^{i,j}}{\partial y^2}+\sigma^{i,j}\chi_m^{i,j}(x,y)\right]dxdy = \frac{1}{h_ih_j}\int_{-h_j/2}^{h_j/2}\int_{-h_i/2}^{h_i/2}[\sigma_s^{i,j}\phi^{i,j}(x,y)+Q^{i,j}]dxdy. \tag{14}$$

If we define the surface averaged neutron currents for the direction m at the boundaries of the node (i, j) :

$$\tilde{J}_{mL}^{i,j}=\frac{1}{h_j}\int_{-h_j/2}^{h_j/2}-\frac{\mu_m^2}{\sigma^{i,j}}\frac{\partial\chi_m^{i,j}}{\partial x}\bigg|_{x=-h_i/2}dy, \quad (15a)$$

$$\tilde{J}_{mR}^{i,j}=\frac{1}{h_j}\int_{-h_j/2}^{h_j/2}-\frac{\mu_m^2}{\sigma^{i,j}}\frac{\partial\chi_m^{i,j}}{\partial x}\bigg|_{x=h_i/2}dy, \quad (15b)$$

$$\tilde{J}_{mB}^{i,j}=\frac{1}{h_i}\int_{-h_i/2}^{h_i/2}-\frac{\eta_m^2}{\sigma^{i,j}}\frac{\partial\chi_m^{i,j}}{\partial y}\bigg|_{y=-h_j/2}dx, \quad (15c)$$

$$\tilde{J}_{mT}^{i,j}=\frac{1}{h_i}\int_{-h_i/2}^{h_i/2}-\frac{\eta_m^2}{\sigma^{i,j}}\frac{\partial\chi_m^{i,j}}{\partial y}\bigg|_{y=h_j/2}dx, \quad (15d)$$

then we may write the node balance equation as

$$-\frac{1}{h_i}(\tilde{J}_{mL}^{i,j}-\tilde{J}_{mR}^{i,j})-\frac{1}{h_j}(\tilde{J}_{mB}^{i,j}-\tilde{J}_{mT}^{i,j}) + \sigma^{i,j}\bar{\chi}_m^{i,j}=\sigma_s^{i,j}\bar{\phi}^{i,j}+Q^{i,j}. \quad (16a)$$

The remaining four equations come from the continuity conditions:

$$\tilde{J}_{mL}^{i,j}-\tilde{J}_{mR}^{i-1,j}=0, \quad (16b)$$

$$\tilde{J}_{mR}^{i,j}-\tilde{J}_{mL}^{i+1,j}=0, \quad (16c)$$

$$\tilde{J}_{mB}^{i,j}-\tilde{J}_{mT}^{i,j-1}=0, \quad (16d)$$

$$\tilde{J}_{mT}^{i,j}-\tilde{J}_{mB}^{i,j+1}=0 \quad (16e)$$

at the node interfaces depicted in Fig. 2.

If we perform the integration in Eq.(15) using the intra nodal flux distribution[Eq.(9)] and their coefficients[Eq.(13)], the surface averaged currents can be expressed as

$$\tilde{J}_{mL}^{i,j}=-\frac{2\mu_m^2}{\sigma^{i,j}h_i}(3\bar{\chi}_m^{i,j}-2\tilde{\chi}_{mL}^{i,j}-\tilde{\chi}_{mR}^{i,j}) \quad (17a)$$

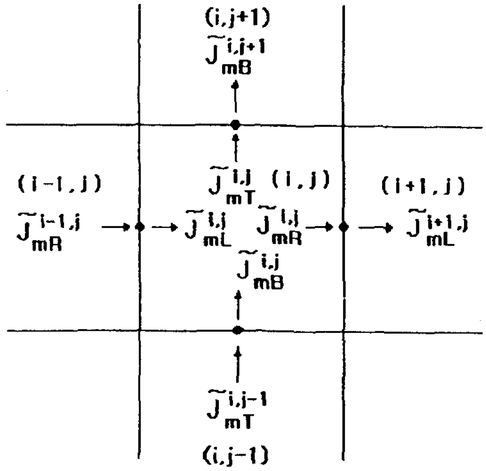

Fig. 2 Current Continuity Conditions

$$\tilde{J}_{mR}^{i,j}=-\frac{2\mu_m^2}{\sigma^{i,j}h_i}(-3\bar{\chi}_m^{i,j}+2\tilde{\chi}_{mR}^{i,j}+\tilde{\chi}_{mL}^{i,j}) \quad (17b)$$

$$\tilde{J}_{mB}^{i,j}=-\frac{2\eta_m^2}{\sigma^{i,j}h_j}(3\bar{\chi}_m^{i,j}-2\tilde{\chi}_{mB}^{i,j}-\tilde{\chi}_{mT}^{i,j}) \quad (17c)$$

$$\tilde{J}_{mT}^{i,j}=-\frac{2\eta_m^2}{\sigma^{i,j}h_j}(-3\bar{\chi}_m^{i,j}+2\tilde{\chi}_{mT}^{i,j}+\tilde{\chi}_{mB}^{i,j}). \quad (17d)$$

By applying Eqs.(17) to Eqs.(16), we finally have the following five equations:

$$-\frac{6\mu_m^2}{\sigma^{i,j}h_i^2}(\tilde{\chi}_{mL}^{i,j}-2\bar{\chi}_m^{i,j}+\tilde{\chi}_{mR}^{i,j}) -\frac{6\eta_m^2}{\sigma^{i,j}h_j^2}(\tilde{\chi}_{mB}^{i,j}-2\bar{\chi}_m^{i,j}+\tilde{\chi}_{mT}^{i,j})+\sigma^{ij}\bar{\chi}_m^{i,j} =\sigma_s^{i,j}\bar{\phi}^{i,j}+Q^{i,j}, \quad (18a)$$

$$-\frac{2\mu_m^2}{\sigma^{i,j}h_i}(3\bar{\chi}_m^{i,j}-2\tilde{\chi}_{mL}^{i,j}-\tilde{\chi}_{mR}^{i,j}) -\frac{2\mu_m^2}{\sigma^{i-1,j}h_{i-1}}(3\bar{\chi}_m^{i-1,j}-2\tilde{\chi}_{mR}^{i-1,j}-\tilde{\chi}_{mL}^{i-1,} \quad (18b)$$

$$-\frac{2\mu_m^2}{\sigma^{i,j}h_i}(3\bar{\chi}_m^{i,j}-2\tilde{\chi}_{mR}^{i,j}-\tilde{\chi}_{mL}^{i,j}) -\frac{2\mu_m^2}{\sigma^{i+1,j}h_{i+1}}(3\bar{\chi}_m^{i+1,j}-2\tilde{\chi}_{mL}^{i+1,j}-\tilde{\chi}_{mR}^{i+1,} \quad (18c)$$

$$-\frac{2\eta_m^2}{\sigma^{i,j}h_j}(3\bar{\chi}_m^{i,j}-2\tilde{\chi}_{mB}^{i,j}-\tilde{\chi}_{mT}^{i,j})-\frac{2\eta_m^2}{\sigma^{i,j-1}h_{j-1}}(3\bar{\chi}_m^{i,j-1}-2\tilde{\chi}_{mT}^{i,j-1}-\tilde{\chi}_{mB}^{i,j-1})=0, \tag{18d}$$

$$-\frac{2\eta_m^2}{\sigma^{i,j}h_j}(3\bar{\chi}_m^{i,j}-2\tilde{\chi}_{mT}^{i,j}-\tilde{\chi}_{mB}^{i,j})-\frac{2\eta_m^2}{\sigma^{i,j+1}h_{j+1}}(3\bar{\chi}_m^{i,j+1}-2\tilde{\chi}_{mB}^{i,j+1}-\tilde{\chi}_{mT}^{i,j+1})=0. \tag{18e}$$

For a node $(i,\ j)$, the node balance equation [Eq.(18a)] consists of a 5-point relation between one node averaged flux and four surface averaged fluxes. Similarly, the surface averaged flux continuity conditions at each interface [Eqs.(18b)-(18e)] consists of 6-point relations, respectively.

4. Analytic Function Expansion Nodal Method for the Simplified Even-Parity Equation

The main idea of AFEN in diffusion theory is to expand the intra nodal scalar flux $\phi^{i,j}(x,y)$ using the analytic solution of the diffusion equation in a node. Fortunately, the diffusion equation is homogeneous(meaning that the right-hand-side of the equation vanishes.) and it is relatively easy to obtain the analytic solution. Unfortunately, however, it is not so simple in SEP to find the analytic solution. This is originated from that the transport equation has nonzero $\phi^{i,j}$ term on the right-hand-side of the equation. Even though we substitute $\phi^{i,j}=\sum_{m=1}^{M}\omega_m\chi_m^{i,j}$ and move it to the left-hand-side, we have to solve for all $\chi_m^{i,j}$'s $(1\le m\le M)$ simultaneously and it would result extreme computational inefficiency. In other words, there is no practical way to convert the transport equation into a homogeneous form. Therefore, we admit that Eq.(7) is nonhomogeneous and the general solution must be a sum of homogeneous solution χ^h_m and a particular solution χ^p_m .(i, j omitted.)

The homogeneous equation corresponding to Eq.(7) is

$$-\frac{\mu_m^2}{\sigma^2}\frac{\partial^2\chi_m^h}{\partial x^2}-\frac{\eta_m^2}{\sigma^2}\frac{\partial^2\chi_m^h}{\partial y^2}+\chi_m^h(x,y)=0. \tag{19}$$

To solve Eq.(19) we use the separation of variables $\chi^h_m(x,y)=X_m(x)\,Y_m(y)$ into Eq.(19) then we have

$$-\frac{\mu_m^2}{\sigma^2}\frac{X_m''}{X_m}-\frac{\eta_m^2}{\sigma^2}\frac{Y_m''}{Y_m}+1=0. \tag{20}$$

Letting

$$\frac{\mu_m^2}{\sigma^2}\frac{X_m''}{X_m}=\alpha^2 \quad\text{and}\quad \frac{\eta_m^2}{\sigma^2}\frac{Y_m''}{Y_m}=\beta^2, \tag{21}$$

where

$$\alpha_k^2+\beta_k^2=1 \tag{22}$$

for infinite number of eigenvalues α_k,β_k (k=1, 2,$\cdots$), we have

$$X_{mk}(x)=C_1\cosh\left(\frac{\sigma\alpha_k x}{\mu_m}\right)+C_2\sinh\left(\frac{\sigma\alpha_k x}{\mu_m}\right) \tag{23a}$$

$$Y_{mk}(x)=C_3\cosh\left(\frac{\sigma\beta_k y}{\eta_m}\right)+C_4\sinh\left(\frac{\sigma\beta_k y}{\eta_m}\right). \tag{23b}$$

Thus, the homogeneous solution for Eq.(19) is

$$\chi_m^h(x,y)=\sum_{k=1}^{\infty}\left[A_k\cosh\left(\frac{\sigma\alpha_k x}{\mu_m}+\frac{\sigma\beta_k y}{\eta_m}\right)+B_k\sinh\left(\frac{\sigma\alpha_k x}{\mu_m}+\frac{\sigma\beta_k y}{\eta_m}\right)\right] \tag{24}$$

with every eigenvalue satisfying Eq.(22).

Now, we are to find the particular solution $\chi_m^p(x,y)$. Reminding that a particular solution can be obtained only when nonhomogeneous terms are known, we need to have the mathematical form of $\phi^{i,j}(x,y)$ [Note that the source term $Q^{i,j}(x,y)$ is always constant within a node.]

The first option is to use arbitrary functions for $\phi^{i,j}(x,y)$. Though many functions such as polynomials, exponential functions, etc. can be candidates, the use of arbitrary functions would result in the extreme computational inefficiency and it may deteriorate the basic idea of AFEN in which the analytic solutions are used for the expansion functions.

The second option is to use the same hyperbolic functions that are already used for $\chi^{h}_{m}(x,y)$ as in Eq.(24). However, in this case, the particular solution $\chi^{p}_{m}(x,y)$ would emerge in the form of the hyperbolic functions and it only adds the same hyperbolic terms to the general solution. Considering that the total number of terms in the expansion function for the intra nodal flux can be easily changed by differing k in Eq.(24), these additional terms contribute nothing in building the mathematical form of the general solution $\chi^{i,j}_{m}(x,y)$.

The third option is what we call the *flat source approximation* in which a constant is used for the approximation to $\phi^{i,j}(x,y)$. In the flat source approximation we do not mean that the scalar flux is always constant in a node. We just use this approximation to obtain the mathematical form of $\chi^{p}_{m}(x,y)$. Even with the flat source approximation, the intra nodal distribution of the scalar flux $\phi^{i,j}(x,y)$ will have the same distribution of $\chi^{i,j}_{m}(x,y)$, since it is calculated by the angular fluxes at each source iteration by Eq.(8).

When $\phi^{i,j}(x,y)$ is assumed to be constant, all the terms in the right-hand-side of Eq.(7) becomes constant, yielding

$$-\frac{\mu_m^2}{\sigma}\frac{\partial^2\chi_m}{\partial x^2}-\frac{\eta_m^2}{\sigma}\frac{\partial^2\chi_m}{\partial y^2}+\sigma\chi_m(x,y)=\text{const}. \quad (25)$$

Then, the particular solution should be

$$\chi^{p}_{m}(x,y)=A_0, \quad (26)$$

where A_0 is a constant. Finally, the general solution becomes

$$\begin{aligned}\chi_m(x,y)&=\chi^{h}_{m}(x,y)+\chi^{p}_{m}(x,y)\\&=\sum_{k=1}^{\infty}\Big[A_k\cosh\Big(\frac{\sigma\alpha_k x}{\mu_m}+\frac{\sigma\beta_k y}{\eta_m}\Big)\\&\quad+B_k\sinh\Big(\frac{\sigma\alpha_k x}{\mu_m}+\frac{\sigma\beta_k y}{\eta_m}\Big)\Big]+A_0.\end{aligned} \quad (27)$$

In the original AFEN method the same constant term as A_0 in Eq.(27) is used for the expansion of the intra nodal scalar flux. We know that the constant can not be an analytic solution of the diffusion equation. [The solution is constant only when $\alpha=\beta=0$ in the equations similar to Eqs.(21) in diffusion formulation, but this violates Eq.(22).] It is interesting that while the constant term is intentionally added in AFEN for the diffusion equation, the constant emerges in the form of the analytic solution in AFEN for the SEP equation.

The number of undetermined coefficient A_0, A_k, B_k, (k=1, 2, ···) in Eq.(27) is determined by the total number of unknowns in a node. Since we have five unknowns per node, two pairs of α_k, β_k are needed and they must satisfy Eq.(22). Taking $\alpha_1=0$, $\beta_1=0$ and $\alpha_2=0$, $\beta_2=0$ and rearranging gives the final form of the expansion function:

$$\begin{aligned}\chi^{i,j}_{m}(x,y)&=C^{i,j}_{m1}+C^{i,j}_{m2}\cosh\Big(\frac{\sigma^{i,j}x}{\mu_m}\Big)+C^{i,j}_{m3}\sinh\Big(\frac{\sigma^{i,j}x}{\mu_m}\Big)\\&\quad+C^{i,j}_{m4}\cosh\Big(\frac{\sigma^{i,j}y}{\eta_m}\Big)+C^{i,j}_{m5}\sinh\Big(\frac{\sigma^{i,j}y}{\eta_m}\Big).\end{aligned} \quad (28)$$

Using the above expansion, the node averaged flux and the surface averaged fluxes defined in Eqs.(10) can be calculated as

$$\begin{aligned}\bar\chi^{i,j}_{m}&=C^{i,j}_{m1}+\frac{2\mu_m}{\sigma^{i,j}h_i}\sinh\Big(\frac{\sigma^{i,j}h_i}{2\mu_m}\Big)C^{i,j}_{m2}\\&\quad+\frac{2\eta_m}{\sigma^{i,j}h_j}\sinh\Big(\frac{\sigma^{i,j}h_j}{2\eta_m}\Big)C^{i,j}_{m4}\end{aligned} \quad (29a)$$

$$\begin{aligned}\tilde\chi^{i,j}_{mL}&=C^{i,j}_{m1}+\cosh\Big(\frac{\sigma^{i,j}h_i}{2\mu_m}\Big)C^{i,j}_{m2}-\sinh\Big(\frac{\sigma^{i,j}h_i}{2\mu_m}\Big)C^{i,j}_{m3}\\&\quad+\frac{2\eta_m}{\sigma^{i,j}h_j}\sinh\Big(\frac{\sigma^{i,j}h_j}{2\eta_m}\Big)C^{i,j}_{m4}\end{aligned} \quad (29b)$$

$$\begin{aligned}\tilde\chi^{i,j}_{mR}&=C^{i,j}_{m1}+\cosh\Big(\frac{\sigma^{i,j}h_i}{2\mu_m}\Big)C^{i,j}_{m2}+\sinh\Big(\frac{\sigma^{i,j}h_i}{2\mu_m}\Big)C^{i,j}_{m3}\\&\quad+\frac{2\eta_m}{\sigma^{i,j}h_j}\sinh\Big(\frac{\sigma^{i,j}h_j}{2\eta_m}\Big)C^{i,j}_{m4}\end{aligned} \quad (29c)$$

$$\tilde{\chi}_{mB}^{i,j}=C_{m1}^{i,j}+\frac{2\mu_m}{\sigma^{i,j}h_i}\sinh\left(\frac{\sigma^{i,j}h_i}{2\mu_m}\right)C_{m2}^{i,j}+\cosh\left(\frac{\sigma^{i,j}h_j}{2\eta_m}\right)C_{m4}^{i,j}-\sinh\left(\frac{\sigma^{i,j}h_j}{2\eta_m}\right)C_{m5}^{i,j} \quad (29d)$$

$$\tilde{\chi}_{mT}^{i,j}=C_{m1}^{i,j}+\frac{2\mu_m}{\sigma^{i,j}h_i}\sinh\left(\frac{\sigma^{i,j}h_i}{2\mu_m}\right)C_{m2}^{i,j}+\cosh\left(\frac{\sigma^{i,j}h_j}{2\eta_m}\right)C_{m4}^{i,j}+\sinh\left(\frac{\sigma^{i,j}h_j}{2\eta_m}\right)C_{m5}^{i,j} \quad (29e)$$

respectively. If we abbreviate as $\bar{x}=\frac{\sigma^{i,j}h_i}{2\mu_m}$ and $\bar{y}=\frac{\sigma^{i,j}h_j}{2\eta_m}$, then Eqs. (29) can be written in the matrix form as

$$\chi_m = A_m\, C_m, \quad (30)$$

where

$$\chi_m=\begin{bmatrix}\bar{\chi}_m^{i,j}\\ \tilde{\chi}_{mL}^{i,j}\\ \tilde{\chi}_{mR}^{i,j}\\ \tilde{\chi}_{mB}^{i,j}\\ \tilde{\chi}_{mT}^{i,j}\end{bmatrix},$$

$$A_m=\begin{bmatrix}1 & \sinh\bar{x}/\bar{x} & 0 & \sinh\bar{y}/\bar{y} & 0\\ 1 & \cosh\bar{x} & -\sinh\bar{x} & \sinh\bar{y}/\bar{y} & 0\\ 1 & \cosh\bar{x} & \sinh\bar{x} & \sinh\bar{y}/\bar{y} & 0\\ 1 & \sinh\bar{x}/\bar{x} & 0 & \cosh\bar{y} & -\sinh\bar{y}\\ 1 & \sinh\bar{x}/\bar{x} & 0 & \cosh\bar{y} & \sinh\bar{y}\end{bmatrix},$$

$$C_m=\begin{bmatrix}C_{m1}^{i,j}\\ C_{m2}^{i,j}\\ C_{m3}^{i,j}\\ C_{m4}^{i,j}\\ C_{m5}^{i,j}\end{bmatrix}$$

The expansion coefficients are obtained by

$$C_m = A_m^{-1}\chi_m, \quad (31)$$

yielding

$$C_{m1}^{i,j}=\frac{\bar{x}\,\bar{y}\cosh\bar{x}\cosh\bar{y}-\sinh\bar{x}\sinh\bar{y}}{(\sinh\bar{x}-\bar{x}\cosh\bar{x})(\sinh\bar{y}-\bar{y}\cosh\bar{y})}\bar{\chi}_m^{i,j}+\frac{\sinh\bar{x}}{2(\sinh\bar{x}-\bar{x}\cosh\bar{x})}(\tilde{\chi}_{mL}^{i,j}+\tilde{\chi}_{mR}^{i,j})+\frac{\sinh\bar{y}}{2(\sinh\bar{y}-\bar{y}\cosh\bar{y})}(\tilde{\chi}_{mB}^{i,j}+\tilde{\chi}_{mT}^{i,j}) \quad (32a)$$

$$C_{m2}^{i,j}=\frac{\bar{x}}{2(\sinh\bar{x}-\bar{x}\cosh\bar{x})}(2\bar{\chi}_m^{i,j}-\tilde{\chi}_{mL}^{i,j}-\tilde{\chi}_{mR}^{i,j}) \quad (32b)$$

$$C_{m3}^{i,j}=\frac{1}{2\sinh\bar{x}}(\tilde{\chi}_{mR}^{i,j}-\tilde{\chi}_{mL}^{i,j}) \quad (32c)$$

$$C_{m4}^{i,j}=\frac{\bar{y}}{2(\sinh\bar{y}-\bar{y}\cosh\bar{y})}(2\bar{\chi}_m^{i,j}-\tilde{\chi}_{mB}^{i,j}-\tilde{\chi}_{mT}^{i,j}) \quad (32d)$$

$$C_{m5}^{i,j}=\frac{1}{2\sinh\bar{y}}(\tilde{\chi}_{mT}^{i,j}-\tilde{\chi}_{mB}^{i,j}). \quad (32e)$$

Applying Eq.(28) and Eqs.(32) to the definitions of the surface average neutron currents in Eqs.(15) gives

$$J_{mL}^{i,j}=\frac{\mu_m}{2}[f_1(\bar{x})^{i,j}(\tilde{\chi}_{mL}^{i,j}-\tilde{\chi}_{mR}^{i,j})+f_2(\bar{x})^{i,j}(2\bar{\chi}_m^{i,j}-\tilde{\chi}_{mL}^{i,j}-\tilde{\chi}_{mR}^{i,j})], \quad (33a)$$

$$J_{mR}^{i,j}=\frac{\mu_m}{2}[f_1(\bar{x})^{i,j}(\tilde{\chi}_{mL}^{i,j}-\tilde{\chi}_{mR}^{i,j})-f_2(\bar{x})^{i,j}(2\bar{\chi}_m^{i,j}-\tilde{\chi}_{mL}^{i,j}-\tilde{\chi}_{mR}^{i,j})], \quad (33b)$$

$$J_{mB}^{i,j}=\frac{\eta_m}{2}[f_1(\bar{y})^{i,j}(\tilde{\chi}_{mB}^{i,j}-\tilde{\chi}_{mT}^{i,j})+f_2(\bar{y})^{i,j}(2\bar{\chi}_m^{i,j}-\tilde{\chi}_{mB}^{i,j}-\tilde{\chi}_{mT}^{i,j})], \quad (33c)$$

$$J_{mT}^{i,j}=\frac{\eta_m}{2}[f_1(\bar{y})^{i,j}(\tilde{\chi}_{mB}^{i,j}-\tilde{\chi}_{mT}^{i,j})-f_2(\bar{y})^{i,j}(2\bar{\chi}_m^{i,j}-\tilde{\chi}_{mB}^{i,j}-\tilde{\chi}_{mT}^{i,j})], \quad (33d)$$

where we use the abbreviations $f_1(\bar{x})^{i,j}=\left(\frac{\cosh\bar{x}}{\sinh\bar{x}}\right)^{i,j}$ and $f_2(\bar{x})^{i,j}=\left(\frac{\bar{x}\sinh\bar{x}}{\sinh\bar{x}-\bar{x}\cosh\bar{x}}\right)^{i,j}$, etc.. Substituting Eqs.(33) into the node balance equation and the surface averaged flux continuity conditions [Eqs.(16)], we finally have

$$-\frac{\mu_m}{h_i}f_2(\bar{x})^{i,j}(2\bar{\chi}_m^{i,j}-\tilde{\chi}_{mL}^{i,j}-\tilde{\chi}_{mR}^{i,j})-\frac{\eta_m}{h_j}f_2(\bar{y})^{i,j}(2\bar{\chi}_m^{i,j}-\tilde{\chi}_{mB}^{i,j}-\tilde{\chi}_{mT}^{i,j})+\sigma^{ij}\bar{\chi}_m^{i,j} = \sigma_s^{i,j}\bar{\phi}^{i,j}+Q^{i,j}, \quad (34a)$$

$$f_1(\bar{x})^{i,j}(\tilde{\chi}_{mL}^{i,j}-\tilde{\chi}_{mR}^{i,j})+f_2(\bar{x})^{i,j}(2\bar{\chi}_m^{i,j}-\tilde{\chi}_{mL}^{i,j}-\tilde{\chi}_{mR}^{i,j})-f_1(\bar{x})^{i-1,j}(\tilde{\chi}_{mL}^{i-1,j}-\tilde{\chi}_{mR}^{i-1,j})+f_2(\bar{x})^{i-1,j}(2\bar{\chi}_m^{i-1,j}-\tilde{\chi}_{mL}^{i-1,j}-\tilde{\chi}_{mR}^{i-1,j})=0, \quad (34b)$$

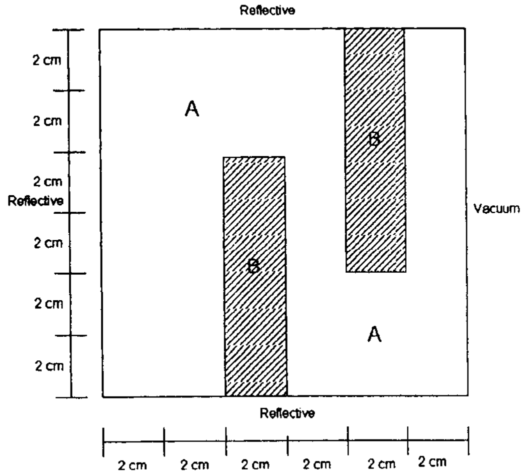

Fig. 3. Configuration of a Sample Problem

$$
\begin{aligned}
&f_1(\bar{x})^{i,j}\left(\tilde{\chi}_{mL}^{i,j}-\tilde{\chi}_{mR}^{i,j}\right)-f_2(\bar{x})^{i,j}\left(2\bar{\chi}_{m}^{i,j}-\tilde{\chi}_{mL}^{i,j}-\tilde{\chi}_{mR}^{i,j}\right)\\
&\quad -f_1(\bar{x})^{i+1,j}\left(\tilde{\chi}_{mL}^{i+1,j}-\tilde{\chi}_{mR}^{i+1,j}\right) \qquad \text{(34c)}\\
&\quad -f_2(\bar{x})^{i+1,j}\left(2\bar{\chi}_{m}^{i+1,j}-\tilde{\chi}_{mL}^{i+1,j}-\tilde{\chi}_{mR}^{i+1,j}\right)=0,
\end{aligned}
$$

$$
\begin{aligned}
&f_1(\bar{y})^{i,j}\left(\tilde{\chi}_{mB}^{i,j}-\tilde{\chi}_{mT}^{i,j}\right)+f_2(\bar{y})^{i,j}\left(2\bar{\chi}_{m}^{i,j}-\tilde{\chi}_{mB}^{i,j}-\tilde{\chi}_{mT}^{i,j}\right)\\
&\quad -f_1(\bar{y})^{i,j-1}\left(\tilde{\chi}_{mB}^{i,j-1}-\tilde{\chi}_{mT}^{i,j-1}\right) \qquad \text{(34d)}\\
&\quad +f_2(\bar{y})^{i,j-1}\left(2\bar{\chi}_{m}^{i,j-1}-\tilde{\chi}_{mB}^{i,j-1}-\tilde{\chi}_{mT}^{i,j-1}\right)=0,
\end{aligned}
$$

$$
\begin{aligned}
&f_1(\bar{y})^{i,j}\left(\tilde{\chi}_{mB}^{i,j}-\tilde{\chi}_{mT}^{i,j}\right)-f_2(\bar{y})^{i,j}\left(2\bar{\chi}_{m}^{i,j}-\tilde{\chi}_{mB}^{i,j}-\tilde{\chi}_{mT}^{i,j}\right)\\
&\quad -f_1(\bar{y})^{i,j+1}\left(\tilde{\chi}_{mB}^{i,j+1}-\tilde{\chi}_{mT}^{i,j+1}\right) \qquad \text{(34e)}\\
&\quad -f_2(\bar{y})^{i,j+1}\left(2\bar{\chi}_{m}^{i,j+1}-\tilde{\chi}_{mB}^{i,j+1}-\tilde{\chi}_{mT}^{i,j+1}\right)=0.
\end{aligned}
$$

The above AFEN difference equation comprises the same matrix equation with differing coefficients. We can expect that the AFEN calculation will take a little longer CPU time than PEN since the process of generating the coefficient matrix is more complicated.

In the PEN and AFEN formulations we have defined five unknowns which consist of one node averaged flux and four surface averaged fluxes. However, among them only three are real unknowns, since the surface average fluxes are shared by the two adjacent nodes. Hence, we solve three equations per node; one for the node averaged flux and two for the vertical and horizontal surface averaged fluxes.

By applying the appropriate boundary conditions such as vacuum and reflective conditions we have the system of linear equations for the SEP fluxes $\chi_m^{i,j}$ and $\tilde{\chi}_{ms}^{i,j}$ ($s=L,B$ or R,T) for the discrete direction $m(1 \le m \le M)$. After solving the system of equations for all the directions, we calculate the scalar flux using Eq.(8) and it will update the right-hand-side of the SEP equation. These processes will be continued until the solution converges.

Table 1. Material Properties of the Problem

Region	σ	σ_s	Q
A	1.0	0.9	1.0
B	1.0	0.0	0.0

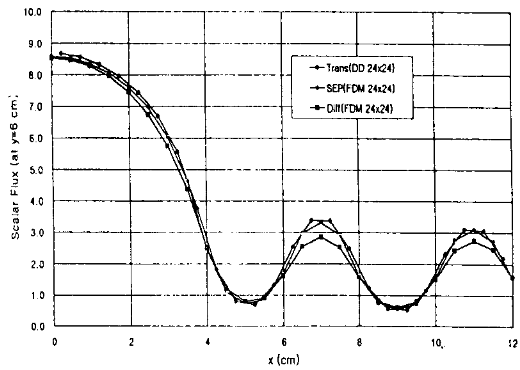

Fig. 4. Transport, SEP, and Diffusion Calculations

5. Numerical Results

In order to confirm how the proposed PEN and AFEN methods work with the SEP transport equation, we perform some numerical tests for a

Table 2. Percent Errors of FDM, PEN, and AFEN for Diffusion Calculations

9.2112		8.7179		7.6413		4.9890		1.0022		2.1881	
9.4850	3.0%	9.1000	4.4%	8.1900	7.2%	6.3990	28.3%	0.7731	22.9%	4.4580	103.7%
9.3680	1.7%	8.9670	2,9%	8.0290	5.1%	5.2640	5.5%	0.9806	2.2%	2.2370	2.2%
9.2300	0.2%	8.7770	0.7%	7.7680	1.7%	4.9890	0.0%	1.0130	1.1%	2.2650	3.5%
8.8733		7.7388		5.8462		4.1909		0.9353		2.1880	
9.2510	4.3%	8.5470	10.4%	6.9000	18.0%	5.9140	41.1%	0.7399	20.9%	4.4630	104.0%
9.0730	2.3%	8.0060	3.5%	6.2720	7.3%	4.3860	4.7%	0.9181	1.8%	2.2590	3.2%
8.9090	0.4%	7.7840	0.6%	5.9380	1.6%	4.1630	0.7%	0.9467	1.2%	2.2660	3.6%
8.3990		6.0414		1.7603		2.8410		0.8659		2.2293	
8.8220	5.0%	7.1920	19.0%	1.2290	30.2%	4.7140	65.9%	0.6877	20.6%	4.5220	102.8%
8.6960	3.5%	6.3190	4.6%	1.7010	3.4%	2.8570	0.6%	0.8057	7.0%	2.3160	3.9%
8.4970	1.2%	6.0880	0.8%	1.7440	0.9%	2.8270	0.5%	0.8947	3.3%	2.3130	3.8%
8.1053		5.4501		1.1529		2.7953		1.3688		2.6140	
8.6080	6.2%	6.8010	24.8%	0.8503	26.2%	4.6840	67.6%	1.0270	25.0%	4.8580	85.8%
8.3850	3.5%	5.6470	3.6%	1.0580	8.2%	2.7860	0.3%	1.3420	2.0%	2.8040	7.3%
8.1700	0.8%	5.4450	0.1%	1.1750	1.9%	2.7760	0.7%	1.3660	0.2%	2.7530	5.3%
7.9994		5.3470		1.2016		3.9947		4.8683		3.8589	
8.5320	6.7%	6.7160	25.6%	0.8908	25.9%	5.7670	44.4%	6.2050	27.5%	6.0930	57.9%
8.2480	3.1%	5.5290	3.4%	1.1620	3.3%	4.1050	2.8%	5.2940	8.7%	4.1500	7.5%
8.0320	0.4%	5.3300	0.3%	1.2080	0.5%	3.9760	0.5%	4.9770	2.2%	4.0260	4.3%
7.9714		5.3307		1.2585		4.6971		6.3902		4.5610	
8.5110	6.8%	6.7020	25.3%	0.9199	26.9%	6.2070	32.1%	7.3810	15.5%	6.5880	44.4%
8.2110	3.0%	5.4910	2.7%	1.2120	3.7%	4.8670	3.6%	6.8050	6.5%	4.8940	7.3%
7.9950	0.3%	5.3100	0.7%	1.2650	0.5%	4.7040	0.1%	6.5410	2.4%	4.7730	4.6%

FDM 48x48 (Reference*)
FDM 6x6 (Relative Error)
PEN 6x6 (〃)
AFEN 6x6 (〃)

* 48x48 values are averaged into 6x6 values
- Average Error : FDM(33.0%), PEN(4.0%), AFEN(1.4%)

sample problem. The computational problem shown in Fig. 3. is 12cm×12cm square domain with the reflective boundary conditions on top, left, and bottom and the vacuum boundary condition on right. It contains a distributed source and pure absorbing regions in it. Table 1 shows the material properties of the regions.

As we mentioned in the preceding section, the SEP is known to be a good approximation to the transport equation. Even though showing the validation of SEP is not a main issue in this paper, we start with providing the transport, SEP, and the diffusion calculations for the sample problem. This will surely give the motivation of the following works. Figure 4 shows the node averaged scalar fluxes calculated by the first-order Boltzmann transport equation, the SEP equation, and the diffusion equation. As spatial differencing schemes, the diamond difference method(DD) is used for the Boltzmann transport equation and the

Table 3. Results of Diffusion Calculations (performed by Pentium 166 MHZ)

	6×6			12×12			24×24		
	No. of Unknowns	Average Error(%)	CPU time (sec)	No. of Unknowns	Average Error(%)	CPU time (sec)	No. of Unknowns	Average Error(%)	CPU time (sec)
FDM	36	33	~0	144	7.7	0.01	576	2.1	0.38
PEN	120	4.0	~0	456	2.3	0.16	1776	1.4	2.47
AFEN	120	1.4	~0	456	1.3	0.22	1776	1.6	2.53

Table 4. Percent Errors of FDM, PEN, and AFEN for SEP Calculations

9.3387		8.8257		7.7367		5.2960		0.9354		2.4773	
9.0300	3.3%	8.2825	6.2%	6.9320	10.4%	3.8153	28.0%	1.2553	34.2%	1.3305	46.3%
9.4250	0.9%	8.9970	1.9%	8.0890	4.6%	5.6260	6.2%	0.7476	20.1%	1.9150	22.7%
8.7230	6.6%	8.1810	7.3%	7.0700	8.6%	4.9760	6.0%	1.0470	11.9%	2.4550	0.9%
9.0152		7.9561		6.0094		4.5910		0.8803		2.4739	
8.5425	5.2%	6.8338	14.1%	4.8413	19.4%	2.7665	39.7%	1.1565	31.4%	1.3208	46.6%
9.1430	1.4%	8.1760	2.8%	6.4240	6.9%	4.8900	6.5%	0.6983	20.7%	1.9240	22.2%
8.3920	6.9%	7.4790	6.0%	5.7460	4.4%	4.3800	4.6%	0.9786	11.2%	2.4600	0.6%
8.5112		6.2745		1.6501		3.2381		0.8138		2.5052	
7.9295	6.8%	5.0473	19.6%	2.2858	38.5%	1.5245	52.9%	1.1025	35.5%	1.3520	46.0%
8.7360	2.6%	6.5220	3.9%	1.5700	4.9%	3.3870	4.6%	0.5947	26.9%	1.9440	22.4%
7.8810	7.4%	6.0810	3.1%	1.8200	10.3%	3.1930	1.4%	0.8898	9.3%	2.5230	0.7%
8.2512		5.7682		1.0846		3.1980		1.2854		2.8218	
7.6220	7.6%	4.3805	24.1%	1.4470	33.4%	1.4875	53.5%	1.7940	39.6%	1.7923	36.5%
8.4710	2.7%	5.9810	3.7%	0.9794	9.7%	3.2900	2.9%	1.0200	20.6%	2.2350	20.8%
7.5450	8.6%	5.5460	3.9%	1.2520	15.4%	3.1140	2.6%	1.3270	3.2%	2.8710	1.7%
8.1672		5.6800		1.1293		4.3973		5.0319		4.0426	
7.5323	7.8%	4.2910	24.5%	1.4680	30.0%	2.6053	40.8%	3.9378	21.7%	3.0698	24.1%
8.3760	2.6%	5.8890	3.7%	1.0620	6.0%	4.4440	1.1%	4.7740	5.1%	3.3500	17.1%
7.3990	9.4%	5.4030	4.9%	1.3010	15.2%	4.0580	7.7%	4.5460	9.7%	3.9120	3.2%
8.1448		5.6673		1.1752		5.0166		6.4982		4.6499	
7.5113	7.8%	4.2848	24.6%	1.5513	32.0%	3.5468	29.3%	5.6850	12.5%	4.1093	11.6%
8.3490	2.5%	5.8630	3.2%	1.0950	6.8%	5.0080	0.2%	6.0180	7.4%	3.8480	17.2%
7.3500	9.8%	5.3690	5.5%	1.3510	15.0%	4.5380	9.5%	5.6170	13.6%	4.4130	5.1%

FDM 48x48 (Reference*)
FDM 6x6 (Relative Error)
PEN 6x6 (〃)
AFEN 6x6 (〃)

* 48x48 values are averaged into 6x6 values
- Average Error : FDM(26.3%), PEN(8.8%), AFEN(7.0%)

finite difference methods(box scheme) are used for both the SEP and the diffusion equation(Diff). We use a relatively fine 24×24 mesh configuration and S_4 quadrature set for all the calculations. The

Table 5. Results of SEP Calculations (performed by Pentium 166 MHZ)

	6×6			12×12			24×24		
	No. of Unknowns	Average Error(%)	CPU time (sec)	No. of Unknowns	Average Error(%)	CPU time (sec)	No. of Unknowns	Average Error(%)	CPU time (sec)
FDM	36	26.3	0.27	144	7.9	1.28	576	2.3	6.15
PEN	120	8.8	0.71	456	4.0	5.93	1776	1.1	76.83
AFEN	120	7.0	1.21	456	3.6	19.44	1776	2.0	84.31

* Without DSA(Diffusion Synthetic Acceleration)

vacuum boundary condition in transport equations is replaced by the Marshak condition in diffusion equation. The result is that while the SEP calculation is in good agreement with the transport calculation everywhere in the domain, the diffusion calculation shows differences near the material interface and the boundaries. This result is rather general and the details on the accuracy of SEP can be found somewhere.[2,3]

The second test is to see the accuracy improvements by PEN and AFEN over FDM in diffusion calculations. Table 2 shows the results of 6×6 calculation for the sample problem. Each box in Table 2 contains the node averaged fluxes by FDM(box scheme), PEN, and AFEN with their percent errors to the FDM 48×48 fine mesh calculation. Table 3 summarizes the results for the different node systems. Examining the accuracies and the elapsed CPU times we notice that PEN and AFEN are superior to FDM and AFEN is slightly better than PEN. This is rather a general result in diffusion theory and we desire to have the similar improvements by PEN and AFEN in the solution of SEP equation.

Finally we examine PEN and AFEN as the computational methods for the SEP equation. Our objective is to achieve the same improvement in SEP calculation. We solve the SEP equation for the sample problem by FDM, PEN and AFEN using the same S_4 quadrature set. The 6×6 node averaged fluxes and their percent errors are shown in Table 4. The reference calculation is the 48×48 FDM as before. As the case of diffusion calculation, the SEP calculation by PEN and AFEN are more accurate than FDM and AFEN slightly shows a better accuracy than PEN. Table 5 summarizes the results of other calculations and it also shows the effectiveness of PEN and AFEN as the nodal methods for SEP equation.

6. Conclusion and Discussion

The purpose of this study is to develope the nodal transport method for the solution of the simplified even-parity equation using the nodal methods developed for the diffusion equation. The motivation is that if we use the discrete-ordinates method for treating the angular variables then SEP has the elliptic differential operator for spatial variables like the diffusion equation. The PEN and AFEN methods are selected as nodal methods since they do not have transverse integration procedures.

In this study we found that the PEN method, which uses polynomials to represent intra nodal flux, can be easily converted to the solution method for SEP. However, the AFEN method in which the analytic solutions are used for the expansion of intra nodal angular flux requires an assumption for the scalar flux distribution in the node. Here, we use a flat source approximation to follow the basic idea of AFEN and to reduce the amount of computations. The numerical results show that the flat source approximation with AFEN works well with the SEP transport calculation.

We conclude that the PEN and AFEN methods have been successfully applied to the nodal solution methods for the SEP transport equation.

In this study PEN and AFEN transport methods are developed by considering only 5 unknowns per node. Even though the smaller number of unknowns does not harm in developing methodology, additional unknowns such as corner fluxes, and surface weighted fluxes are necessary to enhance the overall accuracies of the methods. To speed up the SEP computation in the discrete-ordinates method, the diffusion synthetic acceleration(DSA) is necessary and it would substantially reduce the total number of source iteration. Considering that much effort have been devoted to expand PEN and AFEN methods to the three dimensional diffusion problem including hexagonal geometry, the similar works could be continued in developing the S_N transport methods by selecting the appropriate quadrature set.[10]

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