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A Characteristic Analysis on the Elastic Stiffness of the Tapered-width Leaf Type Holddown Spring Assembly Designed in KOFA's Design Space

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Abstract

An elastic stiffness formula of a leaf type holddown spring(HDS) assembly with a uniformly tapered width from w_0 to w_1 over the length, has been analytically derived based on Euler beam theory and Castigliano's theorem. Elastic stiffnesses of the tapered-width leaf type HDSs(TW-HDSs) designed in the same dimensional design spaces as the KOFA HDSs have been evaluated from the derived formula, in addition, a parametric study on the elastic stiffness of the TW-HDSs has been carried out.

Analysis results show that, as the effects of axial and shear force on the elastic stiffness of the TW-HDSs have been 0.15~0.21% of the elastic stiffness, most of the elastic stiffness is attributed to the bending moment, and that elastic stiffnesses of the TW-HDSs have been about 32~33% higher than those of the KOFA HDSs. It is found that the number of leaves composing a HDS assembly could be lessened by one under the conditions that the TW-HDSs have been adopted in KOFA.

1. Introduction

HDS assemblies, which are attached at the upper most part of each fuel assembly in pressurized water reactors, have the following two main functions ; first, keeping the fuel assembly firmly seated on the lower core plate during normal plant operation with enough holddown force against the buoyancy forces and the upward hydraulic flow forces that act on the fuel assemblies due to normal reactor coolant flow; second, allowing changes to occur in the length of the fuel assembly relative to the space between the upper and lower core plates, while still providing an acceptable holddown force.[1] These changes in relative length can occur due to differential thermal expansion between the fuel assembly and the core support structure, and irradiation induced growth of the fuel assemblies. In case the fuel assembly is lifted-off of the lower core plate and dropped due to insufficient holddown force during even normal plant operation, not only the fuel rods might be dropped, leading to fuel failure, but the structural integrity of the fuel assemblies might be compromised. Therefore in order to maintain the above two main functions under variously incessant loads during normal plant operation, HDS should be designed to have good elastic behavior.[2]

A tapered-thickness leaf type HDS(TT-HDS) adopted in the fuel assemblies for Westinghouse(W) type reactors in Korea, consists of a number of leaves which are bent into design shapes and machined to have uniformly tapered thickness along the leaf length.[3,4,5] Usually, the TT-HDS consists of two or more leaves depending on the holddown force requirements. Although the leaf type HDS has the advantage of maintaining a large holddown force with small spring deflection and requiring small spaces for setting it up, it is known that both the machining of each leaf into design shapes and estimating its characteristics within specific design variables[2] is difficult. So, some nuclear fuel vendors have developed their own methodologies to estimate the TT-HDS's characteristics and have used them for the initial estimates of holddown force. For example, W developed empirical formulas[6,7] for each leaf spring based on actual tests and Siemens/KWU derived a formula by defining the leaf as a horizontal cantilever and applying an Euler beam model[8]. For design purposes, both nuclear vendors directly use the force-deflection curves obtained by testing production springs.

Some research attempting to assess the stiffness characteristics of the TT-HDS have been successful. Kim et al.[9] performed a spring characteristic analysis by ADINA code and Song et al.[10,11] developed a methodology to predict the elastic stiffness of the TT-HDS by Euler beam theory and Castigliano's theorem based on the bending strain energy. Yim et al.[2] carried out a stiffness characteristic analysis and design optimization by ANSYS code. Recently Song[12] performed an elastic stiffness analysis on the TW-HDS and reported that the elastic stiffness of the TW-HDS was higher than that of the TT-HDS.

In this paper we analytically derive an elastic stiffness formula of the TW-HDS based on the total strain energy and also present parametric study results on the elastic stiffness of the TW-HDSs designed in the same dimensional design spaces as KOFA(Kor-

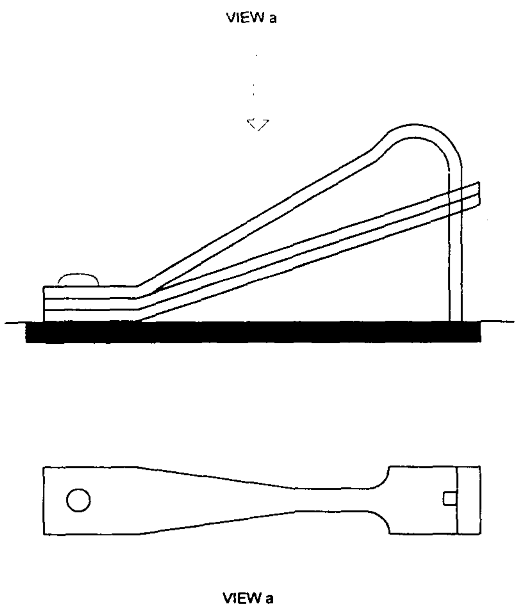

Fig. 1. Leaf Type Holddown Spring Assembly with Tapered Width(TW-HDS)

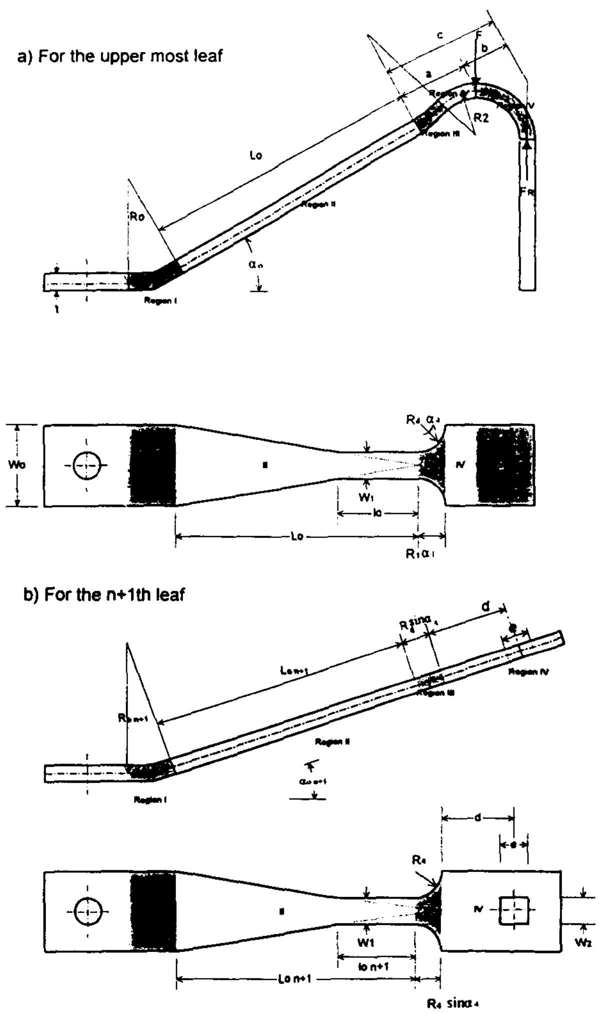

Fig. 2. Design Variables for Each Leaf of TW-HDS

ean Fuel Assembly) HDSs.

2. Derivation of an Elastic Stiffness Formula

2.1. Bending Moments, Shear and Axial Forces for Eeach Leaf

A schematic diagram of the TW-HDS is shown in Figure 1 and its design variables are shown in Figure 2, 3, and 4. For convenience' sake, in this analysis each leaf is divided into 4 or 5 regions as shown in Figure 3 and Figure 4. The bending moments(M_i), shear forces(V_i), axial forces(P_i), and second moments of the beam cross-sectional area(I_i) can be obtained from the free-body diagrams in each region for calculating the total strain energy in each leaf.

2.1.1. For the Top Leaf

2.1.1.1. Region I

$$
\begin{aligned}
M_1 &= -\left\{(L_o+a)\cos\alpha_o+2R_o\sin\frac{1}{2}(\alpha_o-\theta)\cos\theta\right\}F \\
&\quad+\left\{(L_o+c)\cos\alpha_o+2R_o\sin\frac{1}{2}(\alpha_o-\theta)\cos\theta\right\}F_R \\
I_1 &= \frac{1}{12}w_o t^3 \qquad ,0\le\theta\le\alpha_o \qquad (1)\\
P_1 &= (F-F_R)\ \sin(\alpha_o-\theta) \\
V_1 &= (F-F_R)\ \cos(\alpha_o-\theta)
\end{aligned}
$$

2.1.1.2. Region II

$$
\begin{aligned}
M_2 &= -(x+a)F\cos\alpha_o+(x+c)F_R\cos\alpha_o \\
I_2 &= \frac{1}{12}wt^3 \qquad (2\text{-a})\\
P_2 &= (F-F_R)\ \sin\alpha_o \\
V_2 &= (F-F_R)\ \cos\alpha_o \\
&\text{where, } w=\frac{x+R_1\alpha_1}{L_o+R_1\alpha_1}w_o \quad ,\ l_o\le x\le L_o \\
&\qquad\quad\ w=w_1 \quad ,\ 0\le x\le l_o
\end{aligned}
$$

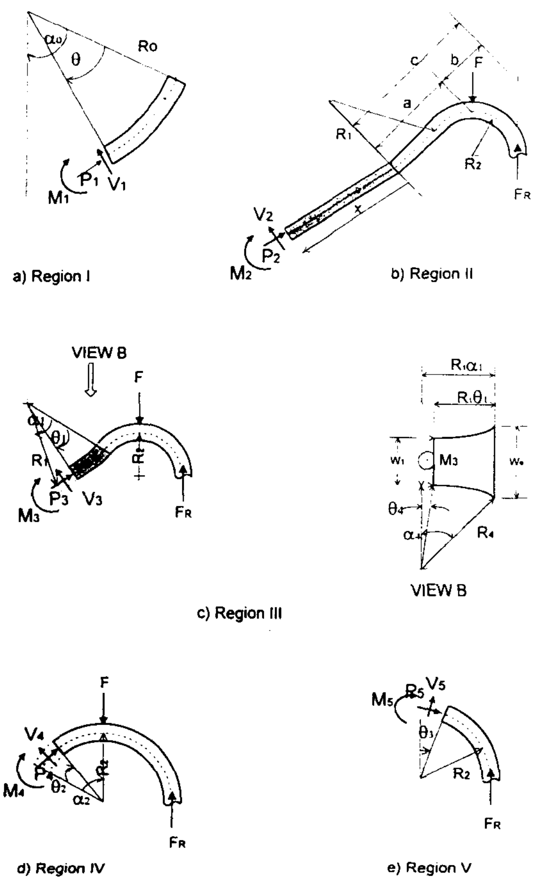

Fig. 3. Free Body Diagram in Each Region for Upper most Leaf of TW-HDS

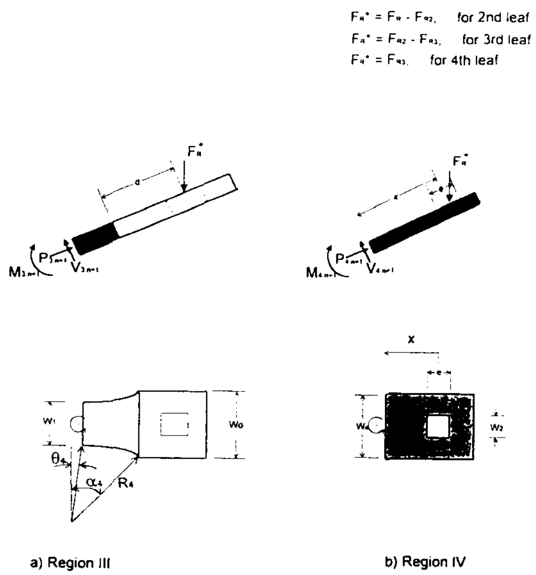

Fig. 4. Free Body Diagram in Region Ⅲ & Ⅳ for n^{+1}th Leaf of TW-HDS

2.1.1.3 Region III

$$M_3 = -F\{a-R_1\sin(\alpha_1-\theta_1)\}\cos\alpha_o + F_R\{c-R_1\sin(\alpha_1-\theta_1)\}\cos\alpha_o$$
$$I_3 = \frac{1}{12}wt^3$$
$$P_3 = (F-F_R)\ \sin(\alpha_o+\alpha_1-\theta_1)$$
$$V_3 = (F-F_R)\ \cos(\alpha_o+\alpha_1-\theta_1) \tag{3}$$

where,

$$w = w_1 + 2R_4(1-\cos\theta_4)$$
$$\theta_4 = \sin^{-1}\frac{R_1}{R_4}(\alpha_1-\theta_1)$$
$$0\le\theta_1\le\alpha_1$$
$$0\le\theta_4\le\alpha_4$$

2.1.1.4. Region IV

$$M_4 = -FR_2\sin(\alpha_2-\theta_2)+F_RR_2\{1+\sin(\alpha_2-\theta_2)\}$$
$$I_4 = \frac{1}{12}w_ot^3 \qquad ,0\le\theta_2\le\alpha_2$$
$$P_4 = (F-F_R)\ \sin(\alpha_2-\theta_2)$$
$$V_4 = (F-F_R)\ \cos(\alpha_2-\theta_2) \tag{4}$$

2.1.1.5. Region V

$$M_5 = -F_RR_2(1-\sin\theta_3)$$
$$I_5 = \frac{1}{12}w_ot^3 \qquad ,0\le\theta_3\le\frac{\pi}{2} \tag{5}$$
$$P_5 = F_R\sin\theta_3$$
$$V_5 = -F_R\cos\theta_3$$

2.1.2. For the Lower(n+1th ; n≥1) Leaf

In region I and II for the lower(n+1th;n≥1) leaf, bending moments, shear forces, axial forces, and second moments of the beam cross-sectional area are expressed the same as equation(1) and (2-a), except for the leaf width for region II, which is expressed as the following equation(2-b). In region III and IV, bending moments, shear forces, axial forces, and second moments of the beam cross-sectional area are expressed as the following equation(6) and (7). And, for each lower leaf, the resultant reaction force(F^*_R) is acting downwards at the reaction point as shown in Figure 4.

2.1.2.1. Region II

$$w = \frac{x+R_4\sin\alpha_4}{L_o+R_4\sin\alpha_4}w_o \qquad ,\ l_o\le x\le L_o \tag{2-b}$$
$$w = w_1 \qquad ,\ 0\le x\le l_o$$

2.1.2.2. Region III

$$M_{3\ n+1} = -F^*_R\{d+R_4(\sin\alpha_4-\sin\theta_4)\}\cos\alpha_o$$
$$I_{3\ n+1} = \frac{1}{12}\{w_1+2R_4(1-\cos\theta_4)\}t^3 \qquad ,0\le\theta_4\le\alpha_4$$
$$P_{3\ n+1} = -F^*_R\sin\alpha_o$$
$$V_{3\ n+1} = -F^*_R\cos\alpha_o \tag{6}$$

2.1.2.3. Region IV

$$M_{4\ n+1} = -F^*_R\,x\ \cos\alpha_o$$
$$I_{4\ n+1} = \frac{1}{12}wt^3$$
$$P_{4\ n+1} = -F^*_R\sin\alpha_o \tag{7}$$
$$V_{4\ n+1} = -F^*_R\cos\alpha_o$$

where,

$$w = w_o \qquad ,\ \frac{e}{2}\le x\le d$$
$$w = w_o - w_2 \qquad ,\ 0\le x\le\frac{e}{2}$$

2.2. Total Strain Energy in Each Leaf

When the TW-HDS is deformed, the total strain energy in each leaf is[13]

$$U_n = \sum_{i=1}^{V}\left\{\int\frac{M_i^2}{2E_iI_i}ds+\int\frac{P_i^2}{2A_iE_i}ds+\int\frac{\tau^2}{2G_i}dV\right\} \tag{8}$$

where,

- U_n : Total starin energy in the nth leaf
- M_i : Bending moment
- E_i : Elastic modulus
- A_i : Cross-sectional area
- P_i : Axial force
- G_i : Shear modulus
- I_i : Second moment of the beam cross-sectional area
- τ : Shear stress

Assuming that the shear stresses are distributed uniformly across the width and solving the equilibrium equations for the plane stress condition,[13] we can obtain the shear-stress distribution in a beam of rectangular cross section as follows.

$$\tau = \frac{V_i}{2I_i}\left[\left(\frac{t_x}{2}\right)^2 - y_1^2 \right] \tag{9}$$

where,

- V_i : shear force
- t_x : distance from the neutral axis to the top surface of a beam
- y_1 : distance from the neutral axis to an arbitrary point on a beam cross section

2.3. In-Line Deflections at Loading(F) and Reaction(F_R) Point

In-line deflections at loading and reaction point are obtained by differentiating the total strain energy (U_n) in each leaf with respect to the load at that point. (Castigliano's theorem[13])

2.3.1. For the Top Leaf

$$\delta_{1F} = \frac{\partial U_1}{\partial F} = AA_1 F - AB_1 F_R \tag{10}$$

$$\delta_{1F_R} = \frac{\partial U_1}{\partial F_R} = -AB_1 F + BB_1 F_R \tag{11}$$

2.3.2. For the Lower Leaf

$$\delta_{2\,F_R} = \frac{\partial U_2}{\partial F_R} = BB_2 (F_R - F_{R2}) \text{ , for the 2nd leaf} \tag{12-a}$$

$$\delta_{3\,F_{R2}} = \frac{\partial U_3}{\partial F_{R2}} = BB_3 (F_{R2} - F_{R3}) \text{ , for the 3rd leaf} \tag{12-a}$$

$$\delta_{4\,F_{R3}} = \frac{\partial U_4}{\partial F_{R3}} = BB_4\ F_{R3} \text{ , for the 4th leaf} \tag{12-c}$$

AA_1, AB_1, BB_1, BB_2, BB_3, and BB_4 are coefficients expressed as a function of design variables in Figure 2 and Figure 3, which are attached in Appendix. And F_R, F_{R2}, and F_{R3} are the reactions at the reaction points of each leaf.

2.4. Constraint Conditions on the In-Line Deflections at Reaction Point

Assuming that the in-line deflections at the reaction point between leaves are equal, we can get constraint conditions as follows.

$$\delta_{1\,F_R} = -\delta_{2\,F_R} \text{ , for the top and 2nd leaf} \tag{13-a}$$

$$\delta_{2\,F_R} = \delta_{3\,F_{R2}} \text{ , for the 2nd and 3rd leaf} \tag{13-b}$$

$$\delta_{3\,F_{R2}} = \delta_{4\,F_{R3}} \text{ , for the 3rd and 4th leaf} \tag{13-c}$$

2.5. Elastic Stiffness Formula

From the in-line deflections of equation(10),(11), (12-a,b,c) and constraint conditions of equation(13-a, b,c) we can obtain an elastic stiffness formula of the TW-HDS as follows.

$$K_{ass} = \frac{1}{\delta_{1F}} = \cfrac{1}{AA_1 - \cfrac{AB_1^2}{BB_1 + \cfrac{1}{\sum_{i=2}^{n} \frac{1}{BB_i}}}} \tag{14}$$

3. Results and Discussions

Based on Euler beam theory and Castigliano's theorem, an elastic stiffness formula of the TW-HDS has been analytically derived and by comparison of the derived formula with an elastic stiffness formula of the TT-HDS[10,11] it is found that the structure of those two formulas are identical except for their coefficients. The reason is that a theory and procedure for the derivation of those two formulas are the same except for their geometric shapes and dimensional data.

Table 1. Dimensional Data of the TW-HDS Designed in the 14×14 Type KOFA Design Space

unit : (mm or degree)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	α_0	R_0	L_0	w_0	t	a	b	c	R_1	R_2	α_2	R_4	w_1	w_2	l_0	d	e
Leaf #1	30.0	17.35	81.0	19.0	4.3	11.0	8.0	19	10	7.15	42	4.75	9.5	5.2	38.125	0	0
Leaf #2	28.5	22.15	77	19.0	4.3	9.0	8.0	17	0	0	0	4.75	9.5	5.2	38.125	10.0	8.0
Leaf #3	28.5	27.15	75	19.0	4.3	9.5	8.5	18	0	0	0	4.75	9.5	5.2	38.125	10.5	8.0

Table 2. Dimensional Data of the TW-HDS Designed in the 17×17 Type KOFA Design Space

unit : (mm or degree)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	α_0	R_0	L_0	w_0	t	a	b	c	R_1	R_2	α_2	R_4	w_1	w_2	l_0	d	e
Leaf #1	30.0	17.35	97.0	19.0	4.3	14.6	7.4	22	23.7	7.15	62.5	4.75	9.5	5.2	46.125	0	0
Leaf #2	26.5	22.15	92	19.0	4.3	15.0	8.0	23	0	0	0	4.75	9.5	5.2	43.625	15.5	8.0
Leaf #3	26.5	27.15	88.5	19.0	4.3	14.5	7.5	22	0	0	0	4.75	9.5	5.2	41.875	15.0	8.0
Leaf #4	26.5	32.15	86.5	19.0	4.3	14.5	8.5	23	0	0	0	4.75	9.5	5.2	40.875	15.0	8.0

Table 3. Comparisons of Elastic Stiffnesses for the TW-HDSs and the KOFA HDSs

unit:(N/mm)

Type	TW-HDS		KOFA HDS
	In the case of only considering the bending moment	In the case of considering the bending moment, shear and axial force	
14×14 type	235.800	235.299(156.495)*	169.777~178.917
17×17 type	177.702	177.443(132.554)*	129.526~136.485

* : In the case that number of leaves are reduced by one

Table 4. Variations of Elastic Stiffness of the TW-HDS vs Leaf Width, w_1

width (mm)	Elastic stiffness (N/mm)			
	N leaves		N-1 leaves	
w_1	14×14/16×16 Type (3 leaves)	17×17 Type (4 leaves)	14×14/16×16 Type (2 leaves)	17×17 Type (3 leaves)
6.5	224.252	167.914	149.104	125.400
7.5	227.663	170.892	151.371	127.626
8.5	231.325	174.052	153.820	129.998
9.5	235.299	177.443	156.495	132.554
10.5	239.648	181.115	159.434	135.331
11.5	244.430	185.115	162.679	138.366
12.5	249.708	189.494	166.273	141.495

Table 1 and Table 2 represent geometric dimensional data of the TW-HDSs designed in the same dimensional design spaces as the 14x14 type and the 17×17 KOFA TT-HDS. Table 3 represents elastic stiffnesses of the TW-HDSs from the equation(14) and the geometric dimensional data of Table 1 and Table 2. Being complicated functions of the design variables of leaf springs, all coefficients in equation (10)~(12) and the elastic stiffness in equation (14) are systematically evaluated from an in-house FORTAN5 program.

Table 3 shows that elastic stiffnesses of the TW-HDSs of which leaf thickness is 4.3mm and leaf width is uniformly tapered from w_0(19.0 mm) to w_l (9.5 mm) are 235.299 N/mm and 177.443 N/mm for the 14×14 type and 17×17 type, respectively. These values are 32~33% higher than test results[10,11] of elastic stiffness of the 14×14 type and 17×17

Table 5. Variations of Elastic Stiffness of the TW-HDS vs Leaf Thickness, *t*

width (mm)	Elastic stiffness (N/mm)			
	N leaves		N-1 leaves	
t (mm)	14×14/16×16 Type (3 leaves)	17×17 Type (4 leaves)	14×14/16×16 Type (2 leaves)	17×17 Type (3 leaves)
3.8	162.468	122.502	108.054	91.511
3.9	175.620	132.422	116.422	98.922
4.0	189.461	142.863	126.007	106.721
4.1	204.010	153.838	135.684	114.920
4.2	219.283	165.360	145.842	123.528
4.3	235.299	177.443	156.495	132.554
4.4	252.075	190.100	167.653	142.010
4.5	269.628	203.344	179.328	151.903
4.6	287.973	217.188	191.531	162.246

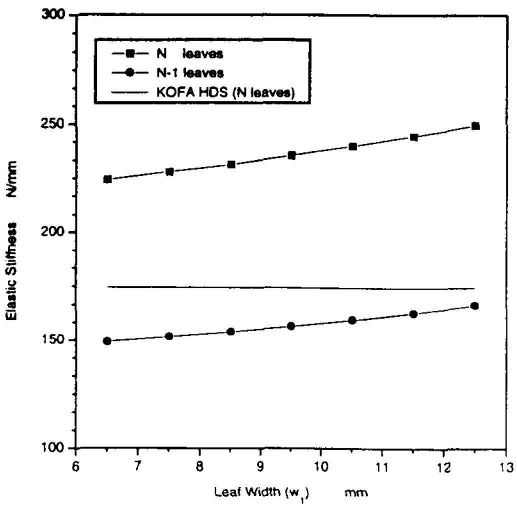

Fig. 5. Variation of Elastic Stiffness vs Leaf Width w_l for 14×14 Type HDS

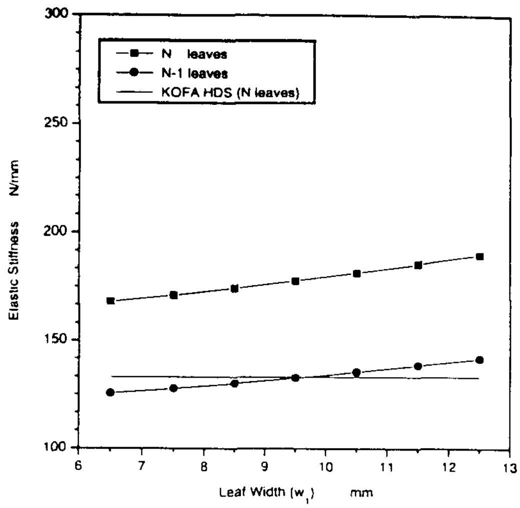

Fig. 6. Variation of Elastic Stiffness vs Leaf Width w_l for 17×17 Type HDS

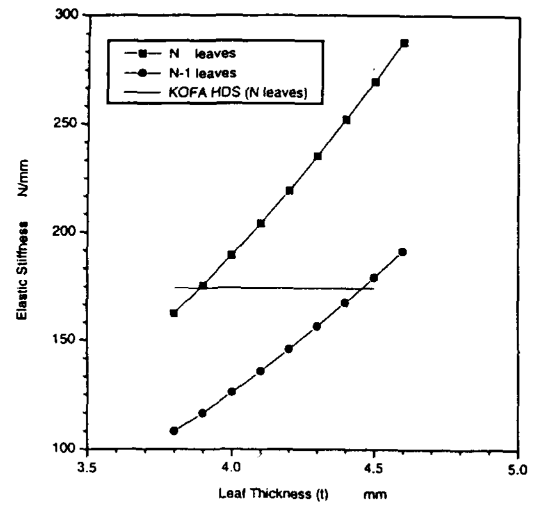

Fig. 7. Variation of Elastic Stiffness vs Leaf Thickness t for 14×14 Type HDS

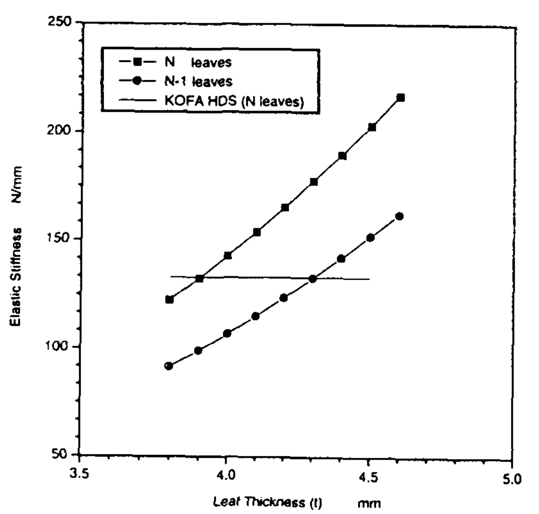

Fig. 8. Variation of Elastic Stiffness vs Leaf Thickness t for 17×17 Type HDS

type KOFA TT-HDS. And Table 3 shows that elastic stiffnesses of the TW-HDSs, the number of leave of which is reduced by one, are comparable with elastic stiffnesses of the 14×14 type and 17×17 type KOFA TT-HDS. In addition, Table 3 shows that, in the case of considering the bending moment, shear and axial forces, elastic stiffnesses are only about 0.15~0.21% lower than those only for considering the bending moment. The reason why elastic stiffnesses are decreasing while additionally considering the axial and shear force is that due to the axial and shear force, the total strain energy is a little more increasing and consequently results in a little more softening the TW-HDS. This implies that the elastic stiffness of the TW-HDS is dominantly attributed to the bending moment. Therefore, in estimating the elastic stiffness of the TW-HDS, it seems reasonable to ignore the effect of shear and axial force.

Table 4 and Figure 5, 6 represent the effect of leaf width(w_l) on elastic stiffnesses of the TW-HDSs with leaf thickness of 4.3 mm. Table 4 and Figure 5, 6 show that the elastic stiffness is in approximately linear correlation with w_l.

Table 5 and Figure 7, 8 represent the effect of leaf thickness(t) on elastic stiffnesses of the TW-HDSs with leaf width(w_l) of 9.5mm. Table 5 and Figure 7, 8 shows that the elastic stiffness is in approxmately cubic correlation with t.

4. Conclusions

We analytically derive the elastic stiffness formula of the TW-HDS and present parametric study results on the elastic stiffness of the TW-HDSs designed in the same dimensional design spaces as the KOFA TT-HDSs. The results from this study are listed as follows.

1. The structure of the elastic stiffness formula of the TW-HDS is identical as that of the TT-HDS except for their coefficients.
2. The effects of shear and axial force on the elastic stiffness of the TW-HDS are only about 0.15~0.21% and most of the elastic stiffness is attributed to the bending moment.
3. Because elastic stiffnesses of the TW-HDSs designed in the same design spaces as the KOFA TT-HDSs are 32~33% higher than those of the KOFA TT-HDSs, elastic stiffnesses of the TW-HDSs, the number of leaves of which is red-

uced by one, are comparable with the test results of elastic stiffnesses of the KOFA TT-HDSs.

4. It would be expected that the cost of the TW-HDSs might be reduced because of easy machinability of the leaf springs and less leaf numbers composing a HDS assembly.
5. The fact that the elastic stiffness are dominantly proportioned to w_i and t^3 implies that the defection of the TW-HDS is mainly caused by the bending moment.

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Appendix

Coefficients in equation(14) are expressed as follows

$$AA_1 = A_1 + B_1 + C_1 + D_1 \tag{A-1}$$

$$AB_1 = A_2 + B_2 + C_2 + D_2 \tag{A-2}$$

$$BB_1 = A_3 + B_3 + C_3 + D_3 + E_3 \tag{A-3}$$

$$BB_{n+1} = A_{3\ n+1} + B_{3\ n+1} + C_{4\ n+1} + G_{3\ n+1} \tag{A-4}$$

$$where, \quad A_n = \frac{12R_n}{Ew_n t^3} \tag{A-5}$$

$$A_1 = A_n\Big[\alpha_n(L_n+a)^2 \cos^2\alpha_n + \frac{8}{3} R_n(L_n+a)\cos\alpha_o(\cos\frac{\alpha_o}{2} - \cos\alpha_n) + R_n^2(\alpha_n - \frac{2}{3}\sin\alpha_n - \frac{1}{6}\sin 2\alpha_n)\Big] + PA1 + VA1 \tag{A-6}$$

$$A_2 = A_n\Big[\alpha_o(L_n+a)(L_n+c)\cos^2\alpha_n + \frac{4}{3} R_n(2L_n+a+c)\cos\alpha_o(\cos\frac{\alpha_o}{2} - \cos\alpha_o) \tag{A-7}$$

$$+R_o^2(\alpha_o-\frac{2}{3}\sin\alpha_o-\frac{1}{6}\sin 2\alpha_o)\Big] + PA2 + VA2$$

$$A_3= A_o\Big[\alpha_o(L_o+c)^2\cos^2\alpha_o+\frac{8}{3}R_o(L_o+c)\cos\alpha_o(\cos\frac{\alpha_o}{2}-\cos\alpha_o)$$
$$+R_o^2(\alpha_o-\frac{2}{3}\sin\alpha_o-\frac{1}{6}\sin 2\alpha_o)\Big] + PA3 + VA3 \quad \text{(A-8)}$$

$$PA1=PA2=PA3=\frac{R_o}{2Ew_ot}(\alpha_o-\frac{1}{2}\sin 2\alpha_o) \quad \text{(A-9)}$$

$$VA1=VA2=VA3=\frac{3R_o}{5Gw_ot}(\alpha_o+\frac{1}{2}\sin 2\alpha_o) \quad \text{(A-10)}$$

$$B_1=B_1(L_o)-B_1(l_o) + PB1 + VB1$$
$$B_2=B_2(L_o)-B_2(l_o) + PB2 + VB2 \quad \text{(A-11,12,13)}$$
$$B_3=B_3(L_o)-B_3(l_o) + PB3 + VB3$$

where, $$B_o=\frac{12(L_o+R_1\alpha_1)\cos^2\alpha_o}{Ew_ot^3} \quad \text{(A-14)}$$

$$B_1(x)= B_o\Big[\frac{1}{2}(x+R_1\alpha_1)^2-2R_1\alpha_1(x+R_1\alpha_1)+(R_1\alpha_1)^2\ln(x+R_1\alpha_1)$$
$$+2a\{x-R_1\alpha_1\ln(x+R_1\alpha_1)\}+a^2\ln(x+R_1\alpha_1)\Big] \quad \text{(A-15)}$$
$$+\frac{12\cos^2\alpha_o}{Ew_1t^3}\frac{1}{3}\{(l_o+a)^3-a^3\}$$

$$B_2(x)= B_o\Big[\frac{1}{2}(x+R_1\alpha_1)^2-2R_1\alpha_1(x+R_1\alpha_1)+(R_1\alpha_1)^2\ln(x+R_1\alpha_1)$$
$$+(a+c)\{x-R_1\alpha_1\ln(x+R_1\alpha_1)\}+ac\ln(x+R_1\alpha_1)\Big] \quad \text{(A-16)}$$
$$+\frac{12\cos^2\alpha_o}{Ew_1t^3}\Big\{\frac{1}{3}l_o^3+\frac{1}{2}(a+c)l_o^2+acl_o\Big\}$$

$$B_3(x)= B_o\Big[\frac{1}{2}(x+R_1\alpha_1)^2-2R_1\alpha_1(x+R_1\alpha_1)+(R_1\alpha_1)^2\ln(x+R_1\alpha_1)$$
$$+2c\{x-R_1\alpha_1\ln(x+R_1\alpha_1)\}+c^2\ln(x+R_1\alpha_1)\Big] \quad \text{(A-17)}$$
$$+\frac{12\cos^2\alpha_o}{Ew_1t^3}\frac{1}{3}\{(l_o+c)^3-c^3\}$$

$$PB1=PB2=PB3= \frac{\sin^2\alpha_o}{Et}\Big[\frac{l_o}{w_1}+\frac{L_o+R_1\alpha_1}{w_o}\ln\Big(\frac{L_o+R_1\alpha_1}{l_o+R_1\alpha_1}\Big)\Big]$$
$$VB1=VB2=VB3= \frac{6\cos^2\alpha_o}{5Gt}\Big[\frac{l_o}{w_1}+\frac{L_o+R_1\alpha_1}{w_o}\ln\Big(\frac{L_o+R_1\alpha_1}{l_o+R_1\alpha_1}\Big)\Big] \quad \text{(A-18,19)}$$

$$C_o=\frac{12R_1\cos^2\alpha_o}{Et^3} \quad \text{(A-20)}$$

$$C_1= C_o\int_0^{\alpha_1}\frac{\{a-R_1\sin(\alpha_1-\theta_1)\}^2}{w_1+2R_4\Big[1-\cos\Big\{\sin^{-1}\frac{R_1}{R_4}(\alpha_1-\theta_1)\Big\}\Big]}d\theta_1 + PC1 + VC1 \quad \text{(A-21)}$$

$$C_2= C_o\int_0^{\alpha_1}\frac{\{a-R_1\sin(\alpha_1-\theta_1)\}\{c-R_1\sin(\alpha_1-\theta_1)\}}{w_1+2R_4\Big[1-\cos\Big\{\sin^{-1}\frac{R_1}{R_4}(\alpha_1-\theta_1)\Big\}\Big]}d\theta_1 + PC2 + VC2 \quad \text{(A-22)}$$

$$C_3= C_o\int_0^{\alpha_1}\frac{\{c-R_1\sin(\alpha_1-\theta_1)\}^2}{w_1+2R_4\Big[1-\cos\Big\{\sin^{-1}\frac{R_1}{R_4}(\alpha_1-\theta_1)\Big\}\Big]}d\theta_1 + PC3 + VC3 \quad \text{(A-23)}$$

$$R_4(1-\cos\alpha_4) = \frac{w_o - w_1}{2}$$

$$\therefore \alpha_4 = \cos^{-1}(1-\frac{w_0-w_1}{2R_4}) \qquad \text{(A-24)}$$

$$R_4 \sin\alpha_4 = R_1\alpha_1$$

$$\alpha_1 = \frac{R_4}{R_1}\sin\left\{\cos^{-1}(1-\frac{w_o-w_1}{2R_4})\right\} \qquad \text{(A-25)}$$

$$PC1 = PC2 = PC3 = \frac{R_1}{Et}\int_0^{\alpha_1}\frac{\sin^2(\alpha_o+\alpha_1-\theta_1)}{w_1+2R_4\,[1-\cos\{\sin^{-1}\frac{R_1}{R_4}(\alpha_1-\theta_1)\}]}d\theta_1 \qquad \text{(A-26)}$$

$$VC1 = VC2 = VC3 = \frac{6R_1}{5Gt}\int_0^{\alpha_1}\frac{\cos^2(\alpha_o+\alpha_1-\theta_1)}{w_1+2R_4\,[1-\cos\{\sin^{-1}\frac{R_1}{R_4}(\alpha_1-\theta_1)\}]}d\theta_1 \qquad \text{(A-27)}$$

$$D_o = \frac{12R_2^3}{Ew_ot^3} \qquad \text{(A-28)}$$

$$D_1 = \frac{1}{2}D_o(\alpha_2-\frac{1}{2}\sin2\alpha_2) + PD1 + VD1 \qquad \text{(A-29)}$$

$$D_2 = D_o(1-\cos\alpha_2) + D_1 + PD2 + VD2 \qquad \text{(A-30)}$$

$$D_3 = D_o\{\alpha_2+2(1-\cos\alpha_2)\} + D_1 + PD3 + VD3 \qquad \text{(A-31)}$$

$$PD1 = PD2 = PD3 = \frac{R_2}{2Etw_o}(\alpha_2-\frac{1}{2}\sin2\alpha_2)$$

$$VD1 = VD2 = VD3 = \frac{3R_2}{5Gtw_o}(\alpha_2+\frac{1}{2}\sin2\alpha_2) \qquad \text{(A-32,33)}$$

$$E_3 = \frac{12R_2^3}{Ew_ot^3}(\frac{3}{4}\pi-2) + PE3 + VE3$$

$$PE3 = \frac{R_2\pi}{4Etw_o} \qquad \text{(A-34,35,36)}$$

$$VE3 = \frac{3R_2\pi}{10Gtw_o}$$

$$C_4 = \frac{12R_4w_o\cos^2\alpha_o}{Ew_ot^3}\int_0^{\alpha_4}\frac{\{d+R_4(\sin\alpha_4-\sin\theta_4)\}^2}{w_1+2R_4(1-\cos\theta_4)}d\theta_4 + PC_3^{\,n+1} + VC_3^{\,n+1} \qquad \text{(A-37)}$$

$$PC_3^{\,n+1} = \frac{R_4\sin^2\alpha_o}{Et}\int_0^{\alpha_4}\frac{1}{[w_1+2R_4(1-\cos\theta_4)]}d\theta_4 \qquad \text{(A-38)}$$

$$VC_3^{\,n+1} = \frac{6R_4\cos^2\alpha_o}{5Gt}\int_0^{\alpha_4}\frac{1}{[w_1+2R_4(1-\cos\theta_4)]}d\theta_4 \qquad \text{(A-39)}$$

$$\alpha_4 = \cos^{-1}(1-\frac{w_o-w_1}{2R_4}) \qquad \text{(A-40)}$$

$$G_3 = \frac{12\cos^2\alpha_o}{Ew_ot^3}\left\{\frac{w_oe^3}{24(w_o-w_2)}+\frac{1}{3}(d^3-\frac{e^3}{8})\right\} + PD_3^{\,n+1} + VD_3^{\,n+1} \qquad \text{(A-41)}$$

$$PD_3^{\,n+1} = \frac{\sin^2\alpha_o}{Et}\left[\frac{e}{2(w_o-w_2)}+\frac{(d-\frac{e}{2})}{w_o}\right] \qquad \text{(A-42)}$$

$$VD_3^{\,n+1} = \frac{6\cos^2\alpha_o}{5Gt}\left[\frac{d-\frac{e}{2}}{w_o}+\frac{e}{2(w_o-w_2)}\right] \qquad \text{(A-43)}$$