Journal of the Korean Nuclear Society
Volume 27, Number 4, August 1995

A Necessary and Sufficient Condition for Multiplicity of Steady-State Solutions of Point-Kinetics Reactor Feedback Systems

Chae Yong Yang
Korea Institute of Nuclear Safety
(Received October 31, 1994)

점동특성시스템이 다중의 정상상태해를 갖기 위한 필요충분조건

양채용
한국원자력안전기술원
(1994. 10. 31 접수)

Abstract

The point-kinetics reactor system which is subject to feedback effects may have multiple steady-state solutions for some operating conditions. A necessary and sufficient condition for multiple steady-state solutions of the point-kinetics reactor feedback system for an external input reactivity is obtained through their theoretical approach. If and only if the steady-state feedback reactivity of the reactor system is not strictly monotonic on some values of the feedback variables, then the reactor system has multiple steady-state solutions for the equilibrium operating conditions corresponding to the values of the feedback variables. Also, if and only if the steady--state feedback reactivity is strictly monotonic on all the feedback variables, then the reactor system has only one steady-state solution for all the operating conditions.

요 약

궤환효과의 지배를 받는 점동특성시스템은 특정 운전조건에서 여러 개의 정상상태해를 가질 수 있다. 점동특성시스템의 다중해 및 비선형시스템의 일반적인 특성과 관련된 이론적 연구를 통하여, 점동특성시스템이 주어진 외부 입력 반응도에 대해 다중의 정상상태해를 갖기 위한 필요충분조건을 얻었다. 즉 정상상태에서의 궤환반응도가 궤환변수의 어떤 값에 대해서 'Strictly Monotonic'이 아니면, 점동특성시스템은 그 값에 대응되는 운전영역에서 다중의 정상상태해를 갖는다. 반면 정상상태에서의 궤환반응도가 궤환변수의 모든 값에 대해서 'Strictly Monotonic'이면, 점동특성시스템은 항상 하나의 정상상태해를 갖는다.

1. Introduction

Feedback effects of an operating system may have significant influence on the dynamic stability of the system. Some design philosophies of the operating system are based on the stability analysis of these feedback effects. The mathematical model that describes a real system considering reactivity feedbacks usually becomes a nonlinear system, which defies an easy analysis. To simplify the analysis of a system one neglects, in many cases, the nonlinear phenomena and linearizes the nonlinear system around a reference point. Then, the linearization theory will be valid only within a limited range of the system variables, and can not be applied beyond the limited range, probably infinitesimal around the reference point. However, the physical system with specific design parameters may be governed by the nonlinear phenomena ; jumping, oscillations, chaos, and etc. due to large perturbations from the reference point. It is, in this case, required that the original nonlinear model of the physical system, without any assumptions, be analyzed.

The analysis of nonlinear system begins with the steady-state analysis of the system, the purpose of which is to find steady-state solutions for given conditions. It is well known that the nonlinear reactor system with positive feedbacks may have multiple steady-state solutions at some reactor parameters [1~4]. Moskalev [1] showed that a reactor model with a positive power coefficient and xenon absorption might have two solutions for a range of parameters. Dean and Chamble [2] analytically obtained multiple solutions for several types of reactors with coolant temperature, void, and xenon effects, and investigated the stability of each solution. They were concerned with multiplicity itself, not its theoretical bases. Cho and Grossman [5] developed a one-dimensional nonlinear feedback model, and Yang and Cho [3] numerically examined the multiplicity of steady-state solutions of the feedback system. Recently, Yang and Cho [4] performed a steady-state analysis and a dynamic-stability analysis of a point-kinetics model incorporating both coolant and fuel temperatures. They also numerically showed that when the coolant temperature coefficient is less than a certain critical value, the number of steady-state solutions is always one, but, when the coolant temperature coefficient is greater than the critical value, the number of solutions may be more than one.

Most of the studies laid emphasis on finding numerical solutions of the feedback models under consideration and then on analyzing the results for their purpose, but not on investigating for what reason the systems have multiple steady-state solutions. This study aims at examining the theoretical basis for the multiplicity of steady-state solutions of the point-kinetics reactor feedback systems. The reactivity of the core can be typically classified by the external reactivity for control of core power and the internal (feedback) reactivity due to feedback effects. For a given external reactivity, the steady-state characteristics of the system, such as the multiplicity of steady-state solutions and their stability, are dependent on the behaviors of the feedback variables and feedback reactivity. Therefore, we can find the design conditions so that the systems avoid operating under the multiple steady-state solutions, by examining the dependency of the internal reactivity on the power levels.

2. Steady-State Solutions of a Point-Kinetics System with Feedbacks

Our basic model for this study is the point-kinetics theory that neutron flux, delay-neutron precursor concentration, and feedback variables are solutions to the equations. Typical feedback variables are fuel and coolant temperatures, and fission-product poisons. The product of reactivity and neutron, $\rho \cdot n$, in the the point-kinetics equation is the main cause for nonlinearity if reactivity is dependent on system variables. Note that steady-state solutions stated in this study signify positive steady-state solutions, except a trivial solution.

2.1. Point-Kinetics Reactor Feedback System

The point-kinetics model considering one-delay -group neutron is written as

$$\frac{dn}{dt} = \frac{\rho(t) - \beta}{\Lambda} n + \lambda C,$$

$$\frac{dC}{dt} = \frac{\beta}{\Lambda} n - \lambda C. \tag{1}$$

The reactivity $\rho(t)$ is determined through changes in specified composition (external control) and/or through changes in coolant and fuel temperatures and fission-product poisons, etc.

A variation of the point kinetics scheme for separation of external and feedback reactivities can be considered. Using separate shape functions, components of reactivity are obtained as [6]

$$\begin{aligned} \rho(t) &= \rho(t, \bar{x}(t)) \\ &= \rho_0(t) + \rho_f(\bar{x}(t)), \end{aligned} \tag{2}$$

where $\rho_0(t)$ is an external control reactivity, e.g., due to control and/or soluble boron, and $\rho_f(\bar{x}(t))$ is a feedback reactivity (or internal reactivity) produced by changes in feedback variables. $\bar{x}$ is a vector consisting of feedback variables, such as fuel and coolant temperatures, fission-product poisons, etc.

Dynamics of the feedback variables is generally given as

$$\frac{d}{dt}\bar{x}(t) = F(n, \bar{x}). \tag{3}$$

The feedback dynamics system Eq. (3) is coupled to the neutron dynamics equation Eq. (1) by the reactivity equation Eq. (2).

In order to calculate the feedback reactivity ρ_f at a specific time, we should include the thermal-hydraulic model and the fission-product poison model into the point-kinetics equation. However, even though the thermal-hydraulic model or fission-product poison model is obtained, it is very difficult to find out temperature and fission-product poison behaviors from the models, because of the strong nonlinearity of system parameters such as thermal-hydraulic parameters. In this study, we deal with a ρ_f itself, not using feedback dynamics equations for ρ_f.

2.2. A Necessary and Sufficient Condition for Multiple Steady-State Solutions

Some studies [1, 4] numerically showed that the point-kinetics system may have multiple steady-state solutions for some operating conditions. We, in this section, investigate the theoretical bases so that the system equation, Eqs. (1) and (3), has multiple steady-state solutions for a constant external reactivity,

We consider an equilibrium core where Eqs. (1) and (3) are in equilibrium, i.e., $\rho = \rho_0 + \rho_f(\bar{x}_s) = 0$ and $C_s = \frac{\beta}{\Lambda\lambda} n_s$, having a steady-state solution (n_s, C_s, $\bar{x}_s$). Total reactivity ρ becomes 0, only if a constant external reactivity ρ_0 is equally balanced by the feedback reactivity $-\rho_f(\bar{x}_s)$. That is,

$$\rho_0 \text{ (a constant value)} = -\rho_f(\bar{x}_s). \tag{4}$$

Eq. (4) indicates that the number of steady-state solutions of the system for a given ρ_0 is entirely dependent on the steady-state characteristics of the feedback reactivity $\rho_f(\bar{x}_s)$. Furthermore, the number of crossing points of $\rho_f(\bar{x}_s)$ function and the constant value $-\rho_0$ is the same as the number of steady-state solutions of the system for the external reactivity ρ_0.

As an example, consider a typical nonlinear feedback reactivity about feedback variables at steady state, $\rho_f(\bar{x}_s)$ such as Figure 1.

When $-\rho_0$ is less than the ρ_1 or greater than the ρ_2 (Case I), the $-\rho_0$ crosses the $\rho_f(\bar{x}_s)$ function at one

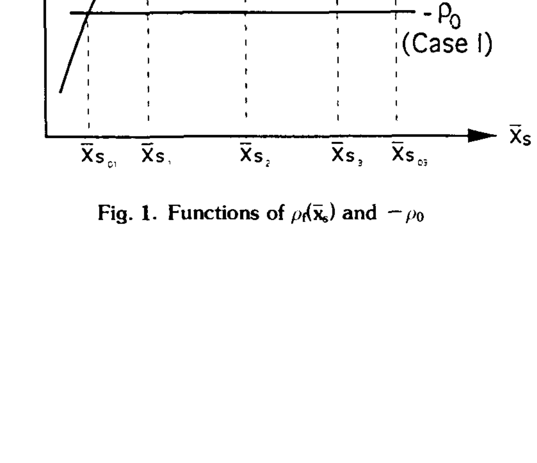

Fig. 1. Functions of $\rho_f(\bar{x}_s)$ and $-\rho_0$

point in a state space, i.e., $\bar{x}_{s01}$ (or $\bar{x}_{s03}$). This means that, for the ρ_0, the number of steady-state solutions of the system is one. When $-\rho_0$ is equal to the ρ_1 or equal to the ρ_2, $-\rho_0$ crosses the $\rho_f(\bar{x}_s)$ function at two points. This means that, for the ρ_0, the number of steady-state solutions is two. When $-\rho_0$ is greater than the ρ_1 and less than the ρ_2 (Case II), the $-\rho_0$ crosses the $\rho_f(\bar{x}_s)$ function at three points in a state space, i.e., $\bar{x}_{s1}$, $\bar{x}_{s2}$, $\bar{x}_{s3}$. This means that, for the ρ_0, the number of steady-state solutions is three.

From the above results, we obtain the following necessary and sufficient condition for the multiplicity of steady-state solutions of the point-kinetics reactor feedback system for an external reactivity. If and only if the steady-state feedback reactivity $\rho_f(\bar{x}_s)$ is not strictly monotonic (1-to-2, 1-to-3, $\cdots$) on some $\bar{x}_s$ [7], the system has multiple steady-state solutions for a range of reactor parameters. Also, if and only if the steady-state feedback reactivity $\rho_f(\bar{x}_s)$ is strictly monotonic (1-to-1) on all $\bar{x}_s$, the system always has one steady-state solution for all the values of the reactor parameters.

3. Applications to Some Reactor Models

We are primarily concerned with temperatures and fission-product poisons as feedback variables. For simple reactor types, feedback models were used for analysis, such as temperature feedback models for short-time behavior of the reactor and xenon feedback models for long-time behavior of the reactor. This section provides the applications to two reactor feedback models from the earlier studies.

3.1. Point-Kinetics Model

Yang and Cho [4] used a point-kinetics model coupled with moderator and fuel temperature feedback dynamics as follows :

$$
\begin{aligned}
\frac{dn}{dt} &= \frac{\rho(T_f, T_m, t) - \beta}{\Lambda} n + \lambda C, \\
\frac{dC}{dt} &= \frac{\beta}{\Lambda} n - \lambda C, \\
\frac{dT_f}{dt} &= \epsilon n - \eta_1 (T_f - T_m), \\
\frac{dT_m}{dt} &= \eta_2 (T_f - T_m) - \eta_3 (T_m - T_a), \qquad (5)
\end{aligned}
$$

where ϵ is the product coefficient of energy and η_i's are the reciprocal constants for heat transfer. The temperature dynamics are coupled to the neutron dynamics by reactivity feedback such as

$$
\rho(T_f, T_m, t) = \rho_0(t) + \int_{\bar{T}_f}^{T_f} \alpha_f dT_f + \int_{\bar{T}_m}^{T_m} \alpha_m dT_m. \qquad (6)
$$

In reality, the temperature coefficients of reactivity are dependent on the temperature itself and hence on the power level. They used the fuel and moderator temperature coefficients given by

$$
\begin{aligned}
\alpha_f(T_f) &= -\frac{4 \times 10^{-4}}{\sqrt{460 + T_f}} \ (/^\circ F), \\
\alpha_m(T_m) &= (-0.167 T_m^2 + 66.7 T_m + k) \times 10^{-8} \ (/^\circ F), \qquad (7)
\end{aligned}
$$

where k is a design parameter representing the moderator temperature coefficient. The greater the value of k is, the larger the moderator temperature coefficient becomes [4].

The steady-state feedback reactivity is expressed as

$$
\begin{aligned}
\rho_f(T_{fs}, T_{ms}) &= \int_{\bar{T}_f}^{T_{fs}} \alpha_f(T_f) dT_f + \int_{\bar{T}_m}^{T_{ms}} \alpha_m(T_m) dT_m \\
&= -8 \times 10^4 \sqrt{460 + T_{fs}} \\
&\quad + \left(\frac{-0.167}{3} T_{ms}^3 + \frac{66.7}{2} T_{ms}^2 + k T_{ms}\right) \\
&\quad + 8 \times 10^4 \sqrt{460 + \bar{T}_f} \\
&\quad - \left(\frac{-0.167}{3} \bar{T}_m^3 + \frac{66.7}{2} \bar{T}_m^2 + k \bar{T}_m\right), \qquad (8)
\end{aligned}
$$

where T_{fs} and T_{ms} are a pair of fuel and moderator temperatures at steady state. Then,

$$\frac{\partial \rho_f(T_{fs},T_{ms})}{\partial T_{ms}} = \frac{-4\times 10^4(1+\frac{\eta_3}{\eta_2})}{\sqrt{(1+\frac{\eta_3}{\eta_2})T_{ms}+(460-\frac{\eta_3}{\eta_2}T_a)}} + (-0.167T_{ms}^2+66.7T_{ms}+k). \quad (9)$$

From Eq. (9), we can analytically obtain the critical value of k at which $\partial\rho_f(T_{fs}, T_{ms})/\partial T_{ms}=0$ has only one solution. When k is less than the critical value, Eq. (8) is strictly monotonic for T_{ms} (or T_{fs}). As an example, when the parameters are given as $\lambda=0.1$, $\beta=0.01$, $\Lambda=1.5\times10^{-5}$, $\in=0.051$, $\eta_1=0.194$, $\eta_2=0.108$, $\eta_3=2.163$, and $T_a=560$ [4], the critical value of k is 3.948×10^4.

If the value of k is less than the critical value, then $\rho_f(T_{fs}, T_{ms})$ is strictly monotonic and increasing on all the values of the steady-state temperatures, and thus the number of steady-state solution of Eq. (5) is always one for all ρ_0. However, if the value of k is greater than the critical value, then $\rho_f(T_{fs}, T_{ms})$ is no longer strictly monotonic and thus the number of the solutions is more than one for some range of ρ_0. Temperature dependences of ρ_f for several values of k are plotted in Figure 2. These results are consistent to the numerical results given in Reference 4.

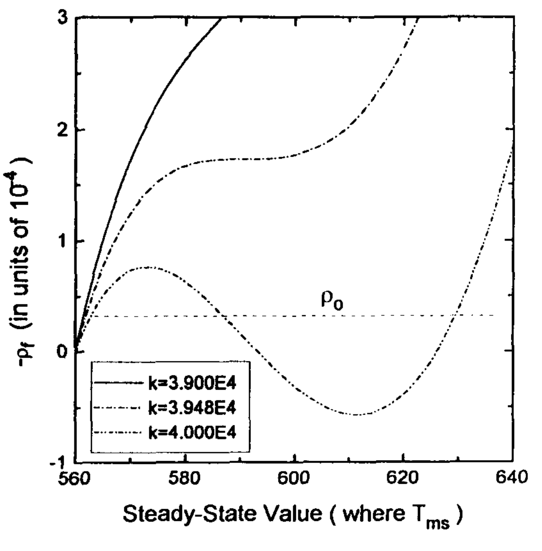

Fig. 2. Feedback Reactivity for Some k

3.2. Space-Dependent Reactor Feedback Model

In this section, we consider a space-dependent model. Cho and Grossman [5] developed the space-time-dependent reactor dynamics equations considering moderator and fuel temperatures, xenon, and soluble boron feedbacks into one-group diffusion equation coupled with the xenon-iodine dynamics equations and energy balance relations in the core. They obtained the steady-state nonlinear reactor equation of the following:

$$\frac{d^2u(x)}{dx^2}+\lambda u = Au\int_0^x u(x')dx' + Bu^2 + C\frac{u^2}{1+u}, \quad 0<x<1, \quad (10)$$

with boundary conditions

$$u(0)=u(1)=0,$$

where $u(x)$ is dimensionless flux at a dimensionless core height x. Feedback constants A, B, and C, and eigenvalue λ are defined in Reference 3.

The right side of Eq. (10) is produced by feedback effects. The first term is a nonlinear term due to the moderator temperature, fuel temperature, and boron feedbacks. The second term is due to the fuel temperature feedback only. The third term is due to the xenon feedback.

Yang and Cho [3] numerically solved Eq. (10) for some sets of A, B, and C. They classified the solution diagrams into three types. Type 1 is a solution diagram consisting of one bifurcation point and two limit points. The solution diagrams show that the system has up to three solutions for some λ. Type 2 is a solution diagram consisting of one bifurcation point and one limit point. The solution diagrams show that the system has up to two solutions for some λ. Type 3 is a solution diagram consisting of one bifurcation point. The solution diagrams show that the system always has only one solution for all the values of λ.

Eq. (10) can be rewritten as

$$\lambda = g(x, u_s(x)) = A\int_0^x u_s(x')dx' + Bu_s + C\frac{u_s}{1+u_s} - \frac{1}{u_s}\frac{d^2}{dx^2}u_s(x), \quad (11)$$

where $u_s(x)$ is a steady-state solution of Eq. (10). The function $g(x, u_s(x))$ can be expressed by the feedback eigenvalue $\lambda_f(x, u_s(x))$ and the external control eigenvalue $\lambda_0(x, u_s(x))$ such as

$$g(x, u_s(x)) = \lambda_f(x, u_s(x)) + \lambda_0(x, u_s(x)), \quad (12)$$

where

$$\lambda_f = A\int_0^x u_s(x')dx' + Bu_s + C\frac{u_s}{1+u_s},$$
$$\lambda_0 = -\frac{1}{u_s}\frac{d^2}{dx^2}u_s(x).$$

If and only if $g(x, u_s(x))$ is not strictly monotonic on some $u_s(x)$, Eq. (10) has multiple steady-state solutions for some values of λ. Also, if and only if $g(x, u_s(x))$ is strictly monotonic on all $u_s(x)$, Eq. (10) always has only one solution for all the values of λ.

It is more meaningful to examine the characteristics of the solutions in the eigenvalue space, which is a plane consisting of λ_0 and λ_f. Define the relationship of λ_0 and λ_f in the $\lambda_0-\lambda_f$ plane for given $u_s(x)$ as the $\lambda_0-\lambda_f$ function. If the system has three steady-state solutions for a range of λ, then the $\lambda_0-\lambda_f$ function is necessarily given as Figure 3 in which the $\lambda_0-\lambda_f$ function crosses the straight line at three points for the range of λ. As a result, the number of the crossing points of the $\lambda_0-\lambda_f$ function and the straight line is equal to the number of steady-state solutions for a given λ.

For $u_s(x)$, we consider a solution diagram for

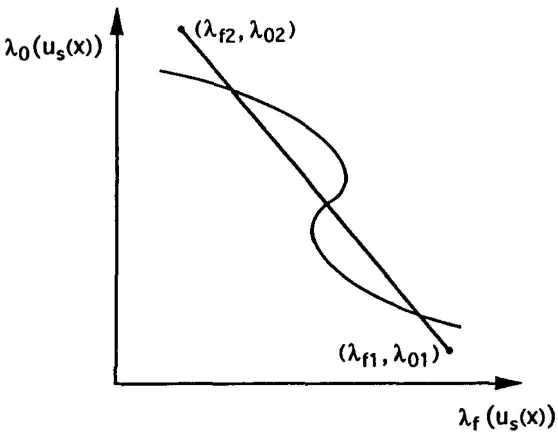

Fig. 3. A λ_0-λ_f Function for Multiple Solutions For a Given λ ; $(\lambda_{f1}+\lambda_{01})=(\lambda_{f2}+\lambda_{02})$

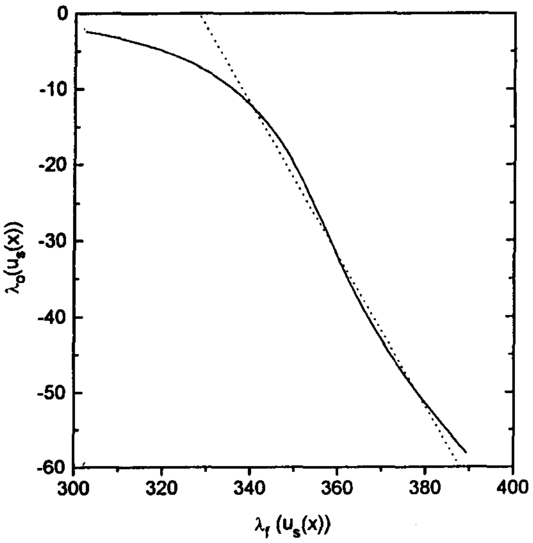

Fig. 4. A λ_0-λ_f Function at x=0.5, and A=−90, B=10, C=509

$A=-90.0$, $B=10.0$, and $C=509.0$, plotted in Figure 8 of Reference 3. The solution diagram shows that the system has three steady-state solutions for $326.696 < \lambda < 329.912$. For the $u_s(0.5)$, the $\lambda_0-\lambda_f$ function is given in Figure 4.

4. Conclusions and Recommendations

We obtained a necessary and sufficient condition for the multiplicity of steady-state solutions of the point-kinetics reactor feedback system for an external reactivity. The reactivity of the core can be typically separated into two components: external reactivity for control of the core power and the internal (feedback) reactivity due to feedback effects. The multiplicity of steady-state solutions of the system is entirely investigated by examining the behaviors of feedback variables and feedback reactivity. If and only if the steady-state feedback reactivity is strictly monotonic for all values of the feedback variables, the reactor system always has one steady-state solution for all the operating conditions. However, if and only if the steady-state feedback reactivity is not strictly monotonic on a range of the feedback variables, the reactor system has multiple steady-state solutions for the operating conditions corresponding to the values of the feedback variables.

The multiplicity of steady-state solutions for an operating condition is caused to the hysteresis effect in system [8]. As a result, it is desired that the feedback reactivity $\rho_f(\bar{x}_s)$ be strictly monotonic on all the values of the feedback variables, $\bar{x}_s$. This is entirely dependent on the design values of moderator temperature coefficient α_m and fuel temperature coefficient α_f. For example, if the value of α_m, for a given α_f, is less than a critical value, $\rho_f(\bar{x}_s)$ is strictly monotonic on all $\bar{x}_s$, but if α_m is beyond the critical value, $\rho_f(\bar{x}_s)$ is no longer strictly monotonic on $\bar{x}_s$.

The reactivity $\rho(t)$ in the point-kinetics equation, which is from the transport equation, is an integrated value of cross sections about the core size. As long as it is known that the point-kinetics reactor feedback equation has multiple steady-state solutions at some operating conditions, the transport equation may also have multiple steady-state solutions for the conditions. We recommend that the multiplicity of the steady-state solutions for the two or three dimensional model of the commercial reactor core be examined for several design values of the temperature-dependent cross sections.

References

1. O.B. Moskalev, "The Possibility of Stationary Neutron Distributions in a Power Reactor," *Zh. Vych. Mat. Fiz.*, **7**, 920 (1967)
2. E.T. Dean and P.L. Chambre, "On the Bifurcation of Solutions of the Nonlinear Eigenvalue Problem $Lu+\lambda b(x)u=g(x, u)$," *SIAM J. Appl. Math.*, **20**, 722 (1971)
3. C.Y. Yang and N.Z. Cho, "A Study of the Multiplicity of Power Distributions in a Nuclear Reactor Subject to Reactivity Feedbacks," *Nucl. Sci. Eng.*, **101**, 243 (1989)
4. C.Y. Yang and N.Z. Cho, "Nonlinear Stability Domain Analysis Applied to the Determination of Operating Margin of a Nuclear Reactor," *Journal of the Korean Nuclear Society*, **24**, 480 (1992)
5. N.Z. Cho and L.M. Grossman, "Optimal Control of Xenon Spatial Oscillations in Load Follow of a Nuclear Reactor," *Nucl. Sci. Eng.*, **83**, 136 (1983)
6. J.B. Yasinsky, "On the Use of Point Kinetics for the Analysis of Rod-Ejection Accidents," *Nucl. Sci. Eng.*, **39**, 241 (1970)
7. R.C. Buck, *Advanced Calculus*, McGraw-Hill Kogakusha, Ltd. (1978)
8. R. Seydel, *From Equilibrium to Chaos*, Elsevier, New York (1988)