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《Original》

General Energy-Dependent Transport Equation with Fission

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Abstract

More detailed calculations of extension to general anisotropic transport equation with fission are studied. These calculations involve that the operator can be splitted into scattering and fission operators when we prove the completeness of general anisotropy.

Applying these operators to the equation makes it easy to extract the slowing-down transient of zero-measure, and completely solves the transport equation.

In addition, the number of the eigenvalues of the second anisotropy is classified with Cs unknown, B_1 and B_2 known constants.

요 약

원자로 내에서 핵 분열이 관계될 때, 일반적인 비 등방성(非等方性) 중성자 수송 방정식의 세부적이고 확장된 계산이 다루어 지고 있다. 우리가 일반 비 등방성인 경우의 해의 완전성을 증명할 때 산란과 분열의 복합 연산자가 각각 산란 연산자와 분열 연산자로 분리 될 수 있다는 것을 보여 주고 있다. 이러한 연산자가 실제 계산에 응용될 때, 해의 완전성에 필요한 측정되지 않은 새로운 항을 끌어낼 수 있고, 이로 말미암아 완전히 방정식을 풀 수 있다. 아울러, 2차 비 등방성의 근의 수가 B_1와 B_2를 기지수로, Cs를 미지수로 하여 자세히 분류되고 있다.

1. Introduction

The solution of neutron transport equation has been greatly developed in recent years. Since Case had been found a complete set of eigenfunctions for the one-speed neutron transport equation for the case of isotropic scattering[1], one-speed anisotropic scattering problem[2], space-angle-energy dependent plane slowing-down problem[3], and anisotropic scattering problem with fission[4] had been completely solved for simple elastic scattering models.

In the case of time-independent neutron transport equation, formerly published papers indicated well how the full-range problems are solved.[2,3,4] For the full-range problems with the boundary conditions specified over the whole μ interval $[-1, +1]$, the coefficients in an eigenfunction expansion are easily found from orthogonality properties. And for the half-range problems where the boundary conditions are specified over the whole μ interval $[-1, 0]$ and $[0, +1]$, the coefficients in an eigenfunction expansion satisfies a singular integral equation. When we consider the coexistence of both fission and slowing-down, such a singular integral equation can be solved in the case of convolution type kernels as Greuling-Goertzel kernels.

In this paper, we will extend the more general anisotropic case with fission, parallel to those developed by Nicolaenko and Zweifel[4].

We deal with some confined aspects of the static energy-dependent neutron transport equation, using the constant cross-section limit and Greuling-Goertzel kernel. Furthermore, we restrict our attention to the case that regeneration occurs only through fission and elastic scattering.

We are primarily concerned with general analysis of Boltzmann equation with fission. And we will classify the number of eigenfunctions of second anisotropic scattering problem. By applying the argument principle to the singular integral equation, we can easily find out the number of roots of regular parts, which is the numerator of the argument.

At the interval $[-1, +1]$, we can assume without proof that the orthogonality theorem is satisfied. Therefore, we shall firstly prove the incompleteness of the given functions, and then insert the slowing-down transient of zero-measure to satisfy the completeness.

2. General Analysis

The general Boltzmann equation with fission[4] is

$$\mu\frac{\partial\Psi}{\partial x}(x,\mu,u)+\Psi(x,\mu,u)$$
$$=\frac{1}{2}C_s\int_{-\infty}^{u}G_o(u-u')du'\int_{-1}^{+1}\Psi(x,\mu,u')d\mu'$$
$$+\frac{1}{2}C_F\chi(u)\int_{-\infty}^{+\infty}du'\int_{-1}^{+1}\Psi(x,\mu',u')d\mu'$$
$$+\frac{1}{2}C_s\Big\{\sum_{n=1}^{\infty}(2n+1)P_n(\mu)\int_{-\infty}^{u}G_n(u-u')du'$$
$$(\times)\int_{-1}^{+1}P_n(\mu')\Psi(x,\mu',u')d\mu'\Big\}+S(x,\mu,u) \quad (2\text{-}1)$$

The $P_n(\mu)$ are Legendre polynomials and $G_n(u)$ the Greuling-Goertzel kernels representing energy-transfer in the n-th angular harmonic[3].

In the case of second anisotropy, the equations are

$$\mu\frac{\partial\Psi}{\partial x}(x,\mu,u)+\Psi(x,\mu,u)$$
$$=\frac{C_s}{2}\int_{-\infty}^{u}G_0(u\text{-}u')du'\int_{-1}^{+1}\Psi(x,\mu',u')d\mu'$$
$$+\frac{C_F}{2}\chi(u)\int_{-\infty}^{+\infty}du'\int_{-1}^{+1}\Psi(x,\mu',u',)d\mu'$$
$$+\frac{3}{2}C_s\mu\int_{-\infty}^{u}G_1(u-u')du'\int_{-1}^{+1}\Psi(x,\mu',u,)\mu'du,$$
$$+\frac{5}{4}C_s(3\mu^2-1)\int_{-\infty}^{+\infty}G_2(u-u')du'\int_{-1}^{+1}\frac{1}{2}(3\mu'^2-1)$$
$$\times\Psi(x,\mu',u')\mu'd\mu'+S(x,\mu,u) \quad (2\text{-}2)$$

By using a Fourier transform of the lethargy

$$\mu\frac{\partial\bar{\Psi}}{\partial x}(x,\mu,k)+\bar{\Psi}(x,\mu,k)$$
$$=\frac{C_s}{2}\bar{G}_o(k)\int_{-1}^{+1}\bar{\Psi}(x,\mu,'k)d\mu'$$
$$+\frac{C_F}{2}\bar{\chi}(k)\int_{-1}^{+1}\bar{\Psi}(x,\mu,'o)d\mu'$$
$$+\frac{3}{2}C_s\mu\bar{G}_1(k)\int_{-1}^{+1}\mu'\bar{\Psi}(x,\mu'k)d\mu'$$
$$+\frac{5}{4}C_s(3\mu^2-1)\bar{G}_2(k)\int_{-1}^{+1}\frac{1}{2}(3\mu'^2-1)\bar{\Psi}(x,\mu,'k)du'$$
$$+\bar{S}(x,\mu,k) \quad (2\text{-}3)$$

where

$$\bar{\Psi}(x,\mu,k)=\int_{-\infty}^{+\infty}\Psi(x,\mu,u')e^{-iku'}du'$$

By setting $k=o$, we find Mika's one-speed equation

$$\mu\frac{\partial\bar{\Psi}}{\partial x}(x,\mu,o)+\bar{\Psi}(x,\mu,o)$$
$$=\frac{(C_F+C_s)}{2}\int_{-1}^{+1}\bar{\Psi}(x,\mu',o)du'$$
$$+\frac{3}{2}C_s\mu B_1\int_{-1}^{+1}\mu'\bar{\Psi}(x,\mu,'o)du'$$
$$+\frac{5}{4}C_s(3\mu^2-1)B_2\int_{-1}^{+1}\frac{1}{2}(3\mu'^2-1)\bar{\Psi}(x,\mu',o)d\mu'$$
$$+\bar{S}(x,\mu,o) \quad (2\text{-}4)$$

where $B_1\equiv\widetilde{G_1}(o)$, $B_2\equiv\widetilde{G_2}(o)$

The solution of Eq. (2-4) is well known[2].

In order to look for normal modes, we substitute the expression

$$\bar{\Psi}(x,\mu,k)=\phi(t,\mu,k)e^{-x/t}$$

into Eq. (2-2) and put

$$\phi_n(t,k)=\int_{-1}^{+1}P_n(\mu)\phi(t,\mu,k)d\mu \quad (2\text{-}5)$$

then, we can write as follows;

$$\phi(t,\mu,k)=\frac{t}{2}P\frac{1}{t-\mu}[C_sG_o(k)\phi_o(t,k)+C_F\bar{\chi}(k)\phi_o(t,o)$$
$$+3\mu C_s\bar{G}_1(k)\phi_1(t,k)$$
$$+\frac{5}{2}(3\mu^2-1)C_s\widetilde{G}_2(k)\phi_2(t,k)]$$
$$+\delta(t-\mu)\lambda(t,k) \quad (2\text{-}6)$$

Integrating both sides of Eq. (2-6) over μ, then we have

$$\lambda(t,k)=\phi_o(t,k)-tC_s\phi_o(t,k)(G_o(k)Q_o(t)$$
$$-tC_F\phi_o(t,o)(\bar{\chi}(k)Q_o(t))$$
$$-3tC_s\phi_1(t,k)(G_1(k)Q_1(t))$$
$$-5tC_s\phi_2(t,k)(G_2(k)Q_2(t)) \quad (2\text{-}7)$$

$Q_o(z)$, $Q_1(z)$, $Q_2(z)$ are Legendre functions of the second kind.

Generally,

$$Q_n(z)=\frac{1}{2}\int_{-1}^{+1}\frac{P_n(\mu)}{z-\mu}d\mu$$

If we assume $\phi_0(t,k)$ is an arbitrary parameter,

$\phi_1(t,k)$ and $\phi_2(t,k)$ may readily be obtained.

Multiplying both sides of Eq. (2-5) by (2m+1) $P_m(\mu)$ and intergating over all μ, and using the relation

$$(2_n+1)\mu P_n(\mu)=(n+1)P_{n+1}(\mu)+nP_{n-1}(\mu),$$

then we obtain the recurrence formula

$$(2m+1)t\phi_m(t,k)-(m+1)\phi_{m+1}(t,k)-m\phi_{m-1}(t,k)$$
$$=tC_s\bar{G}_m(k)\phi_m(t,k)+tC_F\bar{\chi}(k)\phi_o(t,o)\delta_{mo}$$

In the cases of m=0, 1, we have

$$m=0,\quad \phi_1(t,k)=t[\phi_o(t,k)\{1-C_s\bar{G}_o(k)\} -C_F\bar{\chi}(k)\phi_o(t,o)] \tag{2-8}$$

$$m=1,\quad \phi_2(t,k)=\frac{1}{2}[\phi_o(t,k)\{\Big(3-C_sG_1(k)\ (1-C_sG_o(k)\Big)t^2-1)\} -\phi_o(t,o)C_F\{(3-C_s\bar{G}_1(k)\bar{\chi}(k))t^2\}] \tag{2-9}$$

From Eq. (2-7), we look for the discrete regular modes. We notice that there may be four such modes, corresponding to the eigenvalues $\pm L_o$, $\pm L_1$, such that

$$\lambda(\pm L_n,k)=O \tag{2-10}$$

The classification of these eigenvalues will be discussed in the next section.

3. The Number of Discrete Eigenfunctions

The number of discrete eigenfunctions of second anisotropic case will be classified in this section.

Since

$$Q_1(z)=zQ_o(z)-1$$
$$Q_2(z)=\frac{1}{2}(3z^2-1)Q_o(z)-\frac{3}{2}z$$

Substitution of the above formulae into Eq. (2-7) yields,

$$\lambda(t,k)=[\phi_o(t,k)+3tC_s\bar{G}_1(k)\phi_1(t,k) +\frac{15}{2}t^2C_s\bar{G}_2(k)\phi_2(t,k)] -[C_s\bar{G}_o(k)\phi_o(t,k)+C_F\bar{\chi}(k)\phi_o(t,o) +3tC_s\bar{G}_1(k)\phi_1(t,k) +\frac{5}{2}(3t^2-1)C_s\bar{G}_2(k)\phi_2(t,k)]Q_o(t) \tag{2-7}$$

We introduce a new function

$$\Omega(z)=R(z)-zQ_o(z)N(z) \tag{3-1}$$

where $R(z)=\phi_o(z,k)+3zC_s\bar{G}_1(k)\phi_1(z,k) +\frac{15}{2}2z^2C_s\bar{G}_2(k)\phi_2(z,k)$ (3-2)

and $N(z)=C_s\bar{G}_o(k)\phi_o(z,k)+C_F\bar{\chi}(k)\phi_o(z,o) +3zC_s\bar{G}_1(k)\phi_1(z,k) +\frac{5}{2}(3z^2-1)C_s\bar{G}_2(k)\phi_2(z,k)$ (3-3)

The function $\Omega(z)$ is analytic in the whole plane with the cut along the real axis.

Following Mika's procedure of finding the number of discrete eigenfunctions, we set

$$M=\frac{1}{\pi}\ \arg\ \Omega^+(1) \tag{3-4}$$

From Eq. (3-1), we obtain

$$\Omega^{\pm}(t)=R(t)-tQ_o(t)N(t)\pm\frac{\pi i}{2}tN(t), \tag{3-5}$$

$$\arg\ \Omega^+(t)=\tan^{-1}\frac{\pi tN(t)}{2[R(t)-tQ_o(t)N(t)]} \tag{3-6}$$

where $\pm$ means the value taken by an analytic function of z above and below the cut $[-1, +1]$.

We may take arg $\Omega^+(o)=0$.

Since the denominator of Eq. (3-6) has not zero, we shall examine $N(t)$ to look for the number of roots.

And the denominator approaches the negative infinity at $t=1$.

Then we obtain $M\leqq n+1$,

where n is the number of zeros of $N(t)$.

Using Eqs. (2-8) and (2-9), we get

$$N(t)=C_s\phi_o(t,k)[\bar{G}_o(k)+3t^2\bar{G}_1(k)(1-C_s\bar{G}_o(k)) +\frac{5}{4}(3t^2-1)\bar{G}_2(k)\{(3-C_s\bar{G}_1(k)) (1-C_s\bar{G}_o(k))t^2-1\}] +C_F\phi_o(t,o)\bar{\chi}(k)[1-3t^2C_s\bar{G}_1(k) -\frac{5}{4}t^2(3t^2-1)C_s\bar{G}_2(k)(3-C_s\bar{G}_1(k))] \tag{3-7}$$

For the sake of brevity, we set $k=o$ and

$$\phi_o(t,o)=\bar{G}_o(o)=\bar{\chi}(o)\equiv 1$$

therefore, we get

$$N(t)=\frac{15}{4}C_s(3-C_sB_1)(1-C_s-C_F)B_2t^4 +[3B_1C_s(1-C_s-C_F)-\frac{5}{4}C_sB_2(3-C_sB_1) (1-C_s-C_F)-\frac{15}{4}C_sB_2]t^2 +(C_s+\frac{5}{4}C_sB_2+C_F) \tag{3-8}$$

where $B_1=\bar{G}_1(o)$, $B_2=\bar{G}_2(o)$

Let us put $C_s+C_F=\alpha C_s \quad (\alpha\geq 1)$

where α is the parameter which represents the ratio of fission to scattering. The value of α may readily be determined by experimental works.

Then,

$$N(t)=\frac{15}{4}C_s(3-C_sB_1)(1-\alpha C_s)B_2t^4 + C_s[3B_1(1-\alpha C_s)-\frac{5}{4}B_2\{3+(3-C_sB_1)(1-\alpha C_s)\}]t^2 + C_s[\alpha+\frac{5}{4}B_2] \tag{3-9}$$

Now, we introduce

$$f(t)=\frac{4}{C_s}N(t) = 15(3-C_sB_1)(1-\alpha C_s)B_2t^4 + [12B_1(1-\alpha C_s)-5B_2\{3+(3-C_sB_1)(1-\alpha C_s)\}]t^2 + [4\alpha+5B_2] \tag{3-10}$$

Here

$$f(0)=4\alpha+5B_2 \tag{3-11}$$

$$f(1)=10\alpha B_1B_2C_s^2-2\{5B_1B_2+6\alpha B_1+15\alpha B_2\}C_s + 4[\alpha+3B_1+5B_2] \tag{3-12}$$

Since $f(t)$ is quadratic in t^2, we may argue that

if $f(0)\times f(1)<0$, $N(t)$ has only one root at the interval $(0,+1)$

if $f(0)\times f(1)>0$, $N(t)$ has no root, of two roots at $(0, +1)$

From Eq. (3-12), the discriminator D is

$$D=(5B_1B_2+6\alpha B_1+15\alpha B_2)^2 - 40\alpha B_1B_2(\alpha+3B_1+5B_2) = (5B_1B_2-6\alpha B_1-5\alpha B_2)^2 + 80\alpha^2B_1B_2+200\alpha^2B_2^2 \tag{3-13}$$

If B_1 and B_2 have the same sign, D is always positive.

Alternatively, D is rewritten as

$$D=B_1^2(6\alpha-5B_2)^2+10\alpha B_1B_2(14\alpha-5B_2) + 225\alpha^2B_2^2 \tag{3-14}$$

If B_1 is the only unknown constant, the discriminator of Eq. (3-14) is

$$D'=[5\alpha B_2(14\alpha-5B_2)]^2-225\alpha^2B_2^2(6\alpha-5B_2)^2 = -200\alpha^2B_2^2(2\alpha-5B_2)(8\alpha-5B_2) \tag{3-15}$$

Let us examine the following two cases

i) $\frac{5}{2}B_2<\alpha$ or $\frac{5}{8}B_2>\alpha$

$$D'<0.$$

This leads to $D>0$ for all B_1, B_2

ii) $\frac{5}{8}B_2\leqq\alpha\leqq\frac{5}{2}B_2$

$$D'\geqq 0$$

This corresponds to the following two cases:

(a) $\frac{A'-\sqrt{D'}}{(6\alpha-5B_2)^2}>B_1$ or $\frac{A'+\sqrt{D'}}{(6\alpha-5B_2)^2}<B_1$

$$D>0$$

(b) $\frac{A'-\sqrt{D'}}{(6\alpha-5B_2)^2}\leqq B_1\leqq\frac{A'+\sqrt{D'}}{(6\alpha-5B_2)^2}$

$$D\leqq 0,$$

where $A'=-5\alpha B_2(14\alpha-5B_2)$.

For convenience, the results are tabulated in the table.

TABLE

B_2	α	B_1	C_s	M	Remark
$B_2\leqq-\frac{4}{5}$	$\alpha\leqq-\frac{5}{4}B_2$	$B_1>0$	$\frac{A-\sqrt{D}}{10\alpha B_1B_2}<C_s$	3	$B_1=G_1(0)$
			$\frac{A+\sqrt{D}}{10\alpha B_1B_2}>C_s$	1	$B_2=G_2(0)$
			$\frac{A+\sqrt{D}}{10\alpha B_1B_2}\leqq C_s\leqq\frac{A-\sqrt{D}}{10\alpha B_1B_2}$	2	$\alpha=\frac{C_s+C_F}{C_s}$
		$B_1<0$	$\frac{A-\sqrt{D}}{10\alpha B_1B_2}\geqq C_s$	2	$A=5B_1B_2+6\alpha B_1+15\alpha B_2$
			$\frac{A+\sqrt{D}}{10\alpha B_1B_2}\leqq C_s$	2	
			$\frac{A-\sqrt{D}}{10\ B_1B_2}<C_s<\frac{A+\sqrt{D}}{10\alpha B_1B_2}$	1	

(continued)

B_2	α	B_1	C_s	M	Remark
$B_2 > -\frac{4}{5}$	$\alpha > -\frac{5}{4}B_2$	$B_1B_2 > 0$	$\frac{A-\sqrt{D}}{10\alpha B_1B_2} > C_s$	1	
			$\frac{A+\sqrt{D}}{10\alpha B_1B_2} < C_s$	3	
			$\frac{A-\sqrt{D}}{10\alpha B_1B_2} \leqq C_s \leqq \frac{A+\sqrt{D}}{10\alpha B_1B_2}$	2	
	$-\frac{5}{4}B_2 < \alpha < \frac{5}{8}B_2$ or $\alpha > \frac{5}{2}B_2$	$B_1B_2 < 0$	$\frac{A-\sqrt{D}}{10\alpha B_1B_2} \leqq C_s$	2	
			$\frac{A+\sqrt{D}}{10\alpha B_1B_2} \leqq C_s$	2	
			$\frac{A+\sqrt{D}}{10\alpha B_1B_2} < C_s < \frac{A-\sqrt{D}}{10\alpha B_1B_2}$	1	
	$\frac{5}{8}B_2 \leqq \alpha \leqq \frac{5}{2}B_2$	$\frac{A'-\sqrt{D'}}{(5B_2-6\alpha)^2} > B_1$ or $\frac{A'+\sqrt{D'}}{(5B_2-6\alpha)^2} < B_1$	$\frac{A-\sqrt{D}}{10\alpha B_1B_2} \leqq C_s$	2	$A' = -5\alpha B_2$ $(14\alpha - 5B_2)$
			$\frac{A+\sqrt{D}}{10\alpha B_1B_2} \geqq C_s$	2	
			$\frac{A+\sqrt{D}}{10\alpha B_1B_2} < C_s < \frac{A-\sqrt{D}}{10\alpha B_1B_2}$	3	
		$\frac{A'-\sqrt{D'}}{(5B_2-6\alpha)^2} \leqq B_1 \leqq \frac{A'+\sqrt{D'}}{(5B_2-6\alpha)^2}$	all C_s	2	

4. Incompleteness or Completeness of the normal modes

As the former printed case[4], we can state the full-range hypothetical completeness theorem as

$$\Psi(\mu,k) = \sum_n A_{n\pm}\phi(\pm L_n,\mu,k) + \int_{-1}^{+1}\phi(t,\mu,k,)\,dt \quad (4\text{-}1)$$

where

$\Psi(\mu,k)$ is an arbitrary function of μ and of the lethargy-transformed variable k.

In this paper, we are taking $\phi_o(t,k)$ as the unknown expansion coefficient of the continuum modes.

First, we set

$$\begin{aligned}\Psi'(\mu,k) &\equiv \Psi(\mu,k) - \sum_n A_{n\pm}\phi(\pm L_n,\mu,k)\\ &= P\Omega(\mu,k)\\ &\quad + \frac{1}{2}\int_{-1}^{+1} tP\frac{1}{t-\mu}\,[C_s\bar{G}_o(k)\phi_o(t,k)\\ &\quad + C_F\bar{\chi}(k)\phi_o(t,o) + 3\mu C_s\bar{G}_1(k)\phi_1(t,k)\\ &\quad + \frac{5}{2}(3\mu^2-1)C_s\bar{G}_2(k)\phi_2(t,k)]\,dt \qquad (4\text{-}2)\end{aligned}$$

and define

$$B(\mu,k) \equiv \Psi'(\mu,k) - P\Omega(\mu,k) \qquad (4\text{-}3)$$

Splitting the kernel of Eq. (4-3) into singular and regular parts[5],

$$\frac{t}{t-\mu} = \frac{\mu}{t-\mu} + 1 \qquad (4\text{-}4)$$

then regular parts of Eq. (4-3) are

$$\begin{aligned}B_{reg}(\mu,k) &= \frac{1}{2}C_s\bar{G}_o(k)\int_{-1}^{+1}\phi_o(t,k)\,dt\\ &\quad + \frac{1}{2}C_F\bar{\chi}(k)\int_{-1}^{+1}\phi_o(t,o)\,dt\\ &\quad + \frac{3}{2}C_s\bar{G}_1(k)\,\mu\int_{-1}^{+1}\phi_1(t,k)\,dt\\ &\quad + \frac{5}{4}C_s\bar{G}_2(k)\,(3\mu^2-1)\int_{-1}^{+1}\phi_2(t,k)\,dt\end{aligned} \qquad (4\text{-}5)$$

and singular (dominant) parts are

$$\begin{aligned}B_{om}(\mu,k) &= \frac{\mu}{2}\int_{-1}^{+1}P\frac{1}{t-\mu}\,[C_sG_o(k)\phi_o(t,k)\\ &\quad + C_F\bar{\chi}(k)\phi_o(t,o) + 3\mu C_s\bar{G}_1(k)\phi_1(t,k)\\ &\quad + \frac{5}{2}(3\mu^2-1)C_sG_2(k)\phi_2(t,k)\,dt\\ &= \frac{\mu}{2}\int_{-1}^{+1}P\frac{1}{t-\mu}\,[C_sG_o(k)\phi_o(t,k)\end{aligned}$$

$$+C_F\bar{\lambda}(k)\phi_o(t,o)]dt+C(\mu,k) \qquad (4\text{-}6)$$

where

$$C(\mu,k)\equiv\frac{\mu}{2}\int_{-1}^{+1}P\frac{1}{t-\mu}[3\mu C_s\bar{G}_1(k)\phi_1(t,k)+\frac{5}{2}(3\mu^2-1)C_s\bar{G}_2(k)\phi_2(t,k)]dt \qquad (4\text{-}7)$$

Substituting Eqs. (2-8) and (2-9) into Eq. (4-7) and splitting into singular and regular parts, we get

$$\begin{aligned}C_{\rm reg}(\mu,k)=&\frac{3}{2}\mu^2C_s\bar{G}_1(k)\Big[(1-C_s\bar{G}_o(k))\int_{-1}^{+1}\phi_o(t,k)dt\\&-C_F\bar{\lambda}(k)\int_{-1}^{+1}\phi_o(t,o)dt\Big]\\&+\frac{5}{4}\mu(3\mu^2-1)C_s\bar{G}_2(k)\frac{1}{2}(3-C_s\bar{G}_1(k))\\&\times\Big[(1-C_s\bar{G}_o(k))\int_{-1}^{+1}(t+\mu)\phi_o(t,k)dt\\&-C_F\bar{\lambda}(k)\int_{-1}^{+1}(t+\mu)\phi_o(t,o)dt\Big] \qquad (4\text{-}8)\end{aligned}$$

$$\begin{aligned}C_{\rm dom}(\mu,k)=&\frac{3}{2}\mu^3C_s\bar{G}_1(k)\Big[(1-C_s\bar{G}_o(k)\int_{-1}^{+1}P\\&\frac{\phi_o(t,k)}{t-\mu}dt-C_F\bar{\lambda}(k)\int_{-1}^{+1}P\frac{\phi_o(t,0)}{t-\mu}dt\Big]\\&+\frac{5}{4}\mu(3\mu^2-1)C_s\bar{G}_2(k)\Big[\frac{1}{2}\Big\{(3-C_s\bar{G}_1(k))\\&(1-C_s\bar{G}_o(k))\mu^2-1\Big\}\int_{-1}^{+1}P\frac{\phi_o(t,k)}{t-\mu}dt\\&-\frac{1}{2}(3-C_s\bar{G}_1(k))C_F\bar{\lambda}(k)\mu^2\\&\int_{-1}^{+1}P\frac{\phi_o(t,o)}{t-\mu}dt\Big] \qquad (4\text{-}9)\end{aligned}$$

We define

$$\begin{aligned}\bar{\Psi}''(\mu,k)=&\bar{\Psi}'(\mu,k)-B_{\rm reg}(\mu,k)-C_{\rm reg}(\mu,k)\\=&\bar{\Psi}(\mu,k)-\sum_n A_{n\pm}(\pm L_n,\mu,k)\\&-\frac{1}{2}\Big\{C_s\bar{G}_o(k)\int_{-1}^{+1}\phi_o(t,k)dt+C_F\bar{\lambda}(k)\\&\int_{-1}^{+1}t\phi_o(t,o)dt\Big\}\\&-\frac{3}{2}C_s\bar{G}_1(k)\mu\Big\{(1-C_s\bar{G}_o(k))\int_{-1}^{+1}t\phi_o(t,k)dt\\&-C_F\bar{\lambda}(k)\int_{-1}^{+1}\phi_o(t,o)dt\\&-\frac{3}{2}\mu^2C_s\bar{G}_1(k)\Big\{(1-C_s\bar{G}_o(k))\int_{-1}^{+1}\phi_o(t,k)dt\\&-C_F\bar{\lambda}(k)\int_{-1}^{+1}\phi_o(t,o)dt\\&-\frac{5}{4}C_s\bar{G}_2(k)(3\mu^2-1)\Big\{\Big[\int_{-1}^{+1}\phi_o(t,k)[(3\\&-C_s\bar{G}_1(k))(1-C_s\bar{G}_o(k))t^2-1]dt\\&-C_F\bar{\lambda}(k)(3-C_s\bar{G}_1(k))\int_{-1}^{+1}t^2\phi_o(t,o)dt\Big\}\\&-\frac{5}{4}\mu(3\mu^2-1)C_s\bar{G}_2(k)\frac{1}{2}(3-C_s\bar{G}_1(k))\\&\Big\{(1-C_s\bar{G}_o(k))\int_{-1}^{+1}(t+\mu)\phi_o(t,k)dt\\&-C_F\bar{\lambda}(k)\int_{-1}^{+1}(t+\mu)\phi_o(t,o)dt\Big\} \qquad (4\text{-}10)\end{aligned}$$

then, $\bar{\Psi}''(\mu,k)$ has only singular parts.

Also, we define

$$F(z,k)=\frac{1}{2\pi i}\int_{-1}^{+1}\bar{\Psi}''(\mu,k)\frac{d\mu}{\mu-z} \qquad (4\text{-}11)$$

From Eq. (3-1), we can easily calculate the principal value of $\Omega(\mu,k)$.

$$\begin{aligned}P\Omega(\mu,k)=&\phi_o(\mu,k)\Big[1-\mu C_s\bar{G}_o(k)PQ_o(\mu)-3\mu^2C_s\bar{G}_1\\&(k)(1-C_s\bar{G}_o(k))PQ_1(\mu)\\&-\frac{5}{2}\mu C_s\bar{G}_2(k)PQ_2(\mu)\{(3-C_s\bar{G}_1(k))\\&(1-C_s\bar{G}_o(k))\mu^2-1\}\Big]\\&+\phi_o(\mu,o)\Big[-\mu C_F\bar{\lambda}(k)PQ_o(\mu)\\&+3C_F\mu^2C_s\bar{\lambda}(k)\bar{G}_1(k)PQ_1(\mu)\\&+\frac{5}{2}\mu^3C_sC_F(3-C_s\bar{G}_1(k))\bar{\lambda}(k)\bar{G}_2(k)\\&PQ_2(\mu)\Big] \qquad (4\text{-}12)\end{aligned}$$

By inspecting the above equations, we define

$$\begin{aligned}\Omega_s(z,k)=&1-z\Big\{C_s\bar{G}_o(k)Q_o(z)+3zC_s\\&(1-C_s\bar{G}_o(k))\bar{G}_1(k)Q_1(z)\\&+\frac{5}{2}C_s\bar{G}_2(k)\Big[(3-C_s\bar{G}_1(k))\\&(1-C_s\bar{G}_o(k))z^2-1\Big]Q_2(z)\Big\} \qquad (4\text{-}13)\end{aligned}$$

$$\begin{aligned}\Omega_F(z,k)=&-z\Big\{C_F\bar{\lambda}(k)Q_o(z)\\&-3zC_sC_F\bar{\lambda}(k)G_1(k)Q_1(z)\\&-\frac{5}{2}z^2C_s(3-C_sG_1(k))C_F\bar{\lambda}(k)\\&G_2(k)Q_2(z)\Big\} \qquad (4\text{-}14)\end{aligned}$$

Then,

$$\begin{aligned}\Omega_s^{\pm}(\mu,k)=&P\Omega_s(\mu,k)\pm\frac{i\pi\mu}{2}\Big[C_s\bar{G}_o(k)\\&+3C_s\bar{G}_1(k)(1-C_s\bar{G}_o(k))\mu^2\\&+\frac{5}{4}(3\mu^2-1)C_s\bar{G}_2(k)\\&\Big\{(3-C_s\bar{G}_1(k))(1-C_s\bar{G}_o(k))\mu^2-1\Big\}\Big] \qquad (4\text{-}15)\end{aligned}$$

$$\begin{aligned}\Omega_F^{\pm}(\mu,k)=&P\Omega_F(\mu,k)\pm\frac{i\pi\mu}{2}\Big[C_F\bar{\lambda}(k)\\&-3C_s\bar{G}_1(k)C_F\bar{\lambda}(k)\mu^2\\&-\frac{5}{4}\mu^2C_s(3-C_s\bar{G}_1(k))C_F\bar{\lambda}(k)(3\mu^2-1)\Big] \qquad (4\text{-}16)\end{aligned}$$

Also we define

$$N(z,k)=\frac{1}{2\pi i}\int_{-1}^{+1}\frac{\phi_0(t,k)}{t-z}\,dt \tag{4-17}$$

where $\phi_0(t,k)$ is the unknown expansion coefficient.

The functions $N(z,k)$ and $F(z,k)$ are analytic in the z-complex cut from $[-1, +1]$. And $\Omega_s(z,k)$ and $\Omega_F(z,k)$ are also analytic in the z-plane cut from $[-1,+1]$ except some possible poles.

From Eq. (4-17),

$$N^{\pm}(\mu,k)=\pm\frac{1}{2}\phi_0(\mu,k)+\frac{1}{2\pi i}\int_{-1}^{+1}\frac{\phi_0(t,k)}{t-\mu}dt \tag{4-19}$$

First, we get

$$\begin{aligned}
&\Omega_s^+(\mu,k)N^+(\mu,k)-\Omega_s^-(\mu,k)N^-(u,k)\\
&=\phi_0(\mu,k)P\Omega_s\\
&+\frac{\mu}{2}\Big[C_s\bar{G}_0(k)+3C_s\bar{G}_1(k)(1-C_s\bar{G}_0(k))\mu^2\\
&+\frac{5}{4}(3\mu^2-1)C_s\bar{G}_2(k)\{(3-C_s\bar{G}_1(k))\\
&(1-C_s\bar{G}_0(k))\mu^2-1\}\int_{-1}^{+1}\frac{\phi_0(t,k)}{t-\mu}dt
\end{aligned} \tag{4-20}$$

And,

$$\begin{aligned}
&\Omega_F^+(u,k)N^+(\mu,0)-\Omega_F^-(\mu,k)N^-(\mu,0)\\
&=\phi_0(\mu,0)P\Omega_F(u,k)\\
&+\frac{\mu}{2}C_F\bar{\chi}(k)\Big[1-3C_s\bar{G}_1(k)\mu^2\\
&-\frac{5}{4}\mu^2(3\mu^2-1)C_s(3-C_s\bar{G}_1(k))\Big]\\
&(\times)\int_{-1}^{+1}\frac{\phi_0(t,o)}{t-\mu}dt
\end{aligned} \tag{4-21}$$

Adding the above two equations, we get

$$\begin{aligned}
F^+(\mu,k)-F^-(\mu,k)&=\bar{\Psi}''(\mu,k)\\
&=[\Omega_s^+(\mu,k)N^+(\mu,k)-\Omega_s^-(\mu,k)\\
&\quad N^-(\mu,k)]\\
&\quad+[\Omega_F^+(\mu,k)N^+(\mu,0)\\
&\quad-\Omega_F^-(\mu,k)N^-(\mu,0)]
\end{aligned} \tag{4-22}$$

Thus we can write as follows:

$$F(z,k)=\Omega_s(z,k)\;N(z,k)+\Omega_F(z,k)N(z,0) \tag{4-23}$$

We have obtained quite the same formula as that of linear anisotropy by Nicolaenko and Zweifel. Since this equation behaves as fission or slowing-down operator seperately, we can immediately get the solu-tion of plane slowing-down problem in this coexisting case.

Setting k$=o$,

$$N(z,o)=\frac{F(z,o)}{\Lambda(z)} \tag{4-24}$$

where $\Lambda(z)$ is the function used by Case and Mika in the monokinetic case.

And we get

$$N(z,k)=\frac{1}{\Omega_s(z,k)}\left\{F(z,k)-\frac{\Omega_F(z,k)F(z,o)}{\Lambda(z)}\right\} \tag{4-25}$$

From the former definition Eq. (4-17), the analytic function N(z,k) can be found, if N(z,k) have no singularties in z outside the cut $[-1, +1]$. But N(z,k) has some delicate points, i.e., the zeros of $\Lambda(z)$ and $\Omega_s(z,k)$.

First, in case of $\Lambda(\pm L_n)=0\quad(n=0,1)$,

$$F(\pm L_n,0)=0 \tag{4-26}$$

is fulfilled through the discrete modes.

And in the case of $\Omega_s\;[J(k),k]=0$,

$\Omega_s(z,k)$ has zeros. at $z=J(k)$,

This equation leads to the plane slowing down problem, which is solved precisely by Jacobs and McInerney[3].

Following Nicolaenko's procedure, the full-range completeness theorem can be easily extended as

$$\begin{aligned}
\bar{\Psi}(\mu,k)&=\sum_n A_{n\pm}\phi(\pm L_n,\mu,k)+B(k)\bar{\Psi}(J)(,\mu,k)\\
&+\int_{-1}^{+1}\phi(t,\mu,k)\,dt
\end{aligned} \tag{4-27}$$

wher $\bar{\Psi}(J(k),\mu,k)$ are the discrete regular modes of the plane slowing-down problem, as introduced by Jacobs and McInerney. The superposition of these discrete regular modes of the plane slowing-down problem has indeed a null measure, as we called "slowing-down transients."

5. Conclusion

It has been shown that, in the case of the coexistence of elastic scattering and fission with the second anisotropy, splitting of the operators is always possible. When more higher order anisotropy is concerned, it may be possible to have the same operators as these by adding the corresponding higher terms to Ω_s and Ω_F.

Therefore, the extension to general anisotropy may be readily made, but the calculation of finding the roots of higher order is tedious.

More difficult problems lie in some limitations. Previously we restricted our attention to the static state and constant cross-section. In the thermal region, these assumptions can not largely influence on the exact solution. But for the most cases, energy-dependency of the cross-section can never be disregarded and some of the problems derived from the static state, for instance, inelastic scattering problem, must be considered.

If the appropriate kernel dealt with cross-section and some factors related with inelastic scattering are included, the neutron transport equation in time-independent case will be completely solved.

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