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《Original》 Electromagnetic Properties of the Dirac Particles*

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Abstract

A new representation for the Dirac equation, which may be appropriate to describe the interaction of the charged particle with the electric field, is derived by introducing a gauge-independent unitary transformation. It is shown that in this representation the effective Hamiltonian without potentials has a new feature in the non-relativistic limit.

요 약

gauge 獨立한 새로운 unitary 變換을 導入함으로서 振動하고 있는 電場안에서의 spin 1/2 荷電粒子의 運動을 記述하는데 適合한 Dirac 方程式의 表示가 導出되고 있다. 이 새로운 表示에 있어서 potentials를 包含하지 않은 有効 Hamiltonian은 그 非相對論的 極限에서 새로운 特徵을 나타내는 事實을 보여주고 있다.

1. Introduction

The non-relativistic behavior of the spin $\frac{1}{2}$ Dirac particle was studied by Foldy-Wouthuysen[1)] and Tani [2)]. Especially Foldy investigated the properties of the charged Dirac particle with low momentum when moving through the weak, slowly varying, external electromagnetic field (classically describable)[3)]. The extreme relativistic behavior of the Dirac particle was studied by Cini-Touschek[4)] and Pac[5)]. Especially Pac investigated in a synthetical fashion the transformation properties of the Dirac equation in both cases of non-relativistic and extreme-relativistic limits, in connection with the internal symmetry properties of interactions[6)]. Mandelstam studied the gauge-independent and path-dependent transformation properties of the Dirac field in interaction with the electromagnetic field. And he formulated the so-called Mandelstam's electrodynamics without potentials[7)]. Following the Mandelstam's approach mentioned above, Cabbibo and Ferrari dealt with the quantum effects of magnetic monopoles[8)]. Schiff tried to find any relation between quarks and magnetic monopoles, in non-relativistic version of the Mandelstam's approach by modifying the Cabbibo-Ferrari formulation[9)].

In the present paper we introduce a new kind of unitary transformation, which corresponds to a counterpart of the Schiff's formulation in the Mandelstam's approach, to reformulate the Dirac equation with the electromagnetic field in terms of the physical observables, such as the intensity of electric field **E** and the magnetic induction **B**. It is

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shown that the transformation is gauge-independent and path-independent in contrast to the Schiff's case. Especially we are interested in the electromagnetic properties of the transformed new scheme in the non-relativistic limit. For this purpose we transform the scheme to the classical representation (the non-relativistic limit) by the Foldy-Wouthuysen version. It is found that the effective Hamiltonian in the classical representation of the scheme takes a new feature in terms retained to order $(\hbar/mc)^2$ relative to the rest energy of the particle.

In the second section, the new representation of the Dirac equation without potentials is obtained by introducing the new kind of unitary transformation. In the third section, the non-relativistic behavior of the above scheme is discussed, with the aid of the Foldy-Wouthuysen transformation. The last section is left for conclusion.

2. New representation for the Dirac equation

The Dirac equation that describes the motion of a particle with mass m and the charge q in the potentials $\mathbf{A}$ $(\mathbf{r}, t)$, ϕ $(\mathbf{r}, t)$ is

$$i\hbar\frac{\partial\Psi}{\partial t}=H\Psi, \tag{2.1}$$

where

$$H=\left\{q\phi(\mathbf{r}, t)+\frac{c\hbar}{i}\alpha\cdot\left(\nabla-\frac{iq}{c\hbar}\mathbf{A}(\mathbf{r},t)\right)+\beta mc^2\right\} \tag{2.2}$$

and the electromagnetic fields are defined by

$$\mathbf{B}=\nabla\times\mathbf{A}, \tag{2.3}$$

$$\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}-\nabla\phi. \tag{2.4}$$

We now are interested in the case that the intensity of the electric field $\mathbf{E}$ of propagation vector $\mathbf{k}$ and frequency $\omega=c\ |\mathbf{k}|$ is described by

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0\ exp\ \{i(\mathbf{k}\cdot\mathbf{r}-\omega t)\}$$
$$+\mathbf{E}_0{}^*exp\ \{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)\}, \tag{2.5}$$

in which $\mathbf{E}_0$ is a constant complex vector, normal to the direction of propagation $\mathbf{k}$. We define a new field

$$\mathbf{E}'(\mathbf{r},t)=\int_t^\infty \mathbf{E}(\mathbf{r},t')\,dt'. \tag{2.6}$$

Then, from Eqs. (2.4) and (2.5) we have

$$\mathbf{E}'(\mathbf{r},t)=-\frac{i}{\omega}\Big[\mathbf{E}_0 exp\left\{i(\mathbf{k}\cdot\mathbf{r}-\omega t)\right\}$$
$$-\mathbf{E}_0{}^*exp\left\{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)\right\}\Big] \tag{2.7}$$

and

$$\mathbf{B}(\mathbf{r},t)=\frac{c}{\omega}\Big[(\mathbf{k}\times\mathbf{E}_0)\ exp\left\{i(\mathbf{k}\cdot\mathbf{r}-\omega t)\right\}$$
$$+(\mathbf{k}\times\mathbf{E}_0{}^*)\ exp\left\{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)\right\}\Big]$$

or

$$\mathbf{B}(\mathbf{r},t)=\frac{c}{\omega}(\mathbf{k}\times\mathbf{E}) \tag{2.8}$$

where

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}. \tag{2.9}$$

Thus the magnetic induction $\mathbf{B}$ $(\mathbf{r}, t)$ can be rewritten as

$$\nabla\times\mathbf{E}'=\frac{1}{c}\quad, \tag{2.10}$$

provided that the magnetic induction $\mathbf{B}$ vanishes at infinity in time.

Let us now introduce such a unitary transformation*

$$\mathrm{U}=exp\Big[\frac{q}{i\hbar}\int_t^\infty \phi(\mathbf{r},t')\,dt'\Big], \tag{2.11}$$

which is path-independent and satisfies the relation

$$\mathrm{U}^{-1}=\mathrm{U}^{+}=exp\Big[\frac{-q}{i\hbar}\int_t^\infty \phi(\mathbf{r},t')\,dt'\Big]. \tag{2.12}$$

As is well known, the first and second kind of gauge transformations

$$\Psi\to\Psi_0=\ exp\Big[-\frac{iq}{\hbar c}\ \Lambda\ (\mathbf{r},t)\Big], \tag{2.13}$$

$$\left.\begin{aligned}&\mathbf{A}\to\mathbf{A}_0=\mathbf{A}+\nabla\Lambda\\&\phi\to\phi_0=\phi-\frac{1}{c}\frac{\partial\Lambda}{\partial t}\end{aligned}\right\} \tag{2.14}$$

leave the fields $\mathbf{E}$, $\mathbf{E}'$, $\mathbf{B}$ and the form of the wave equation (2.1) unchanged.

Application of Eq. (2.5) to Ψ gives

$$\Psi'=\mathrm{U}\Psi$$
$$=\Psi exp\Big[\frac{q}{i\hbar}\int_t^\infty \phi\ \ (\mathbf{r},t')\,dt'\Big] \tag{2.15}$$

Substitution from Eqs. (2.13) and (2.14) shows that

$$\Psi'\to\Psi_0 exp\Big[\frac{q}{i\hbar}\int_t^\infty\phi_0(\mathbf{r},t')\ \ dt'\Big]$$

* If we define the Mandelstam's path-dependent and guage-independent transformation,

$$M(x)=\exp\Big[\frac{q}{i}\int_x^\infty A_\mu(x)\,dx_\mu\Big],$$

where $x=(\mathbf{r},t)$ and $A_\mu=(\mathbf{A},i\phi)$, then in the frame $dx=(0,0,0,\ idt)$, we get Eq. (2.11).

$$=\Psi_0\; exp\left[-\frac{q}{i\hbar}\int_t^\infty \phi(\mathbf{r},t')dt'\right]$$
$$\times exp\left[-\frac{q}{i\hbar c}\int_t^\infty \frac{\partial \Lambda}{\partial t'}dt'\right]$$
$$=\Psi\;\; exp\left[-\frac{q}{i\hbar}\int_t^\infty \phi(\mathbf{r},t')\, dt'\right]$$
$$=\Psi',$$

provided that the gauge function Λ vanishes at infinity in time. Thus the transformed wave function Ψ' is gauge-independent as in the case of Mandelstam's quantum electrodynamics without potentials.

Equation (2.11) gives for the derivatives of U

$$i\hbar\frac{\partial U^{-1}}{\partial t}=U^{-1}q\phi, \tag{2.16}$$

$$\nabla U^{-1}=U^{-1}\left(\frac{-q}{i\hbar}\right)\int_t^\infty \nabla\phi\;(\mathbf{r},t')dt'$$
$$=U^{-1}\left(\frac{-q}{i\hbar}\right)\int_t^\infty \left\{-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t'}-\mathbf{E}\right\}dt'$$
$$=U^{-1}\left(\frac{iq}{\hbar c}\mathbf{A}+\frac{q}{i\hbar}\mathbf{E}'\right), \tag{2.17}$$

provided that both $\mathbf{A}(\mathbf{r},t)$ and $\phi(\mathbf{r},t)$ vanish at infinity in time.

The wave equation for Ψ' may then be obtained from Eq. (2.1) with the help of Eqs. (2.16) and (2.17):

$$i\hbar\frac{\partial\Psi}{\partial t}=H'\Psi', \tag{2.18}$$

where

$$H'=UHU^{-1} \tag{2.19}$$

and in fact

$$H'=\frac{c\hbar}{i}\alpha\cdot\left(\nabla-\frac{iq}{\hbar}\mathbf{E}'\right)+\beta mc^2, \tag{2.20}$$

which is, of course, gauge-independent. Equations (2.18) and (2.20), without potentials, show that this new representation may be appropriate to describe the interaction of the charged Dirac particle with the electric field as expressed in Eq. (2.5).

In this representation, we may see that the time derivative of $(\mathbf{p}-q\;\mathbf{E}')$ is

$$\overline{(\mathbf{p}-q\mathbf{E}')}^{\bullet}_x=\frac{i}{\hbar}[H',p_x]+\frac{i}{\hbar}[H',-q\mathrm{E}'_x]-q\frac{\partial \mathrm{E}_x}{\partial_t}$$
$$=q\;[\alpha\times\mathbf{B}]_x+qE_x,$$

and then

$$\overline{(\mathbf{p}-q\mathbf{E}')}^{\bullet}=q\{\mathbf{E}+\alpha\times\mathbf{B}\}, \tag{2.21}$$

provided that $\mathbf{E}$ vanishes at infinity in time. Equation (2.21) shows that the time derivative of $(\mathbf{p}-q\mathbf{E})$ may have the analogue of Lorentz force.[10]

3. Application to the non-relativistic limit

As was demonstrated by Foldy and Wouthuysen[1], the electromagnetic properties of the Dirac particle may be exhibited in a very direct way by transforming the Dirac equation into a new representation in which states of positive and negative energy for the particle are separately represented by two-component wave functions. So the Foldy-Wouthuysen representation is paricularly useful for the discussion of the non-relativistic limit of the Dirac equation, in connection with the Pauli representation.

If the Foldy-Wouthuysen transformation is carried out on Eq. (2.1), the Dirac equation takes the form

$$i\hbar\frac{\partial\Psi}{\partial t}=\left\{\beta mc^2+\frac{\beta}{2m}(\mathbf{p}-\frac{q}{c}\mathbf{A})^2+q\phi\right.$$
$$-\frac{q\hbar}{2mc}\beta\sigma\cdot\mathbf{B}$$
$$+\frac{q\hbar}{8m^2c^2}[\sigma\cdot(\mathbf{p}-\frac{q}{c}\mathbf{A})\times\mathbf{E}-\sigma\cdot\mathbf{E}$$
$$\left.\times(\mathbf{p}-\frac{q}{c}\mathbf{A})]-\frac{q\hbar^2}{8m^2c^2}\nabla\cdot\mathbf{E}+\cdots\right\}\Psi, \tag{3.1}$$

where Ψ is the gauge-dependent Foldy-Wouthuysen representation (the classical representation).

If the same transformation is carried out on Eq. (2.18), the Dirac equation in our case takes the form

$$i\hbar\;\frac{\partial\Psi'}{\partial t}=\left\{\beta mc^2+\frac{\beta}{2m}(\mathbf{p}-\frac{q}{c}\mathbf{E}')-\frac{q\hbar}{2mc}\beta\sigma\cdot\mathbf{B}\right.$$
$$+\frac{q\hbar}{8m^2c^2}\;[\sigma\cdot(\mathbf{p}\times\mathbf{E})-\sigma\cdot(\mathbf{E}\times\mathbf{p})]$$
$$\left.-\frac{q\hbar^2}{8m^2c^2}\nabla\cdot\mathbf{E}+\cdots\right\}\;\Psi', \tag{3.2}$$

where Ψ' is the gauge-independent Foldy-Wouthuysen representation. In Eq. (3.2), the effective Hamiltonian without potentials is, of course, gauge-independent. In the derivation of both Eqs. (3.1) and (3.2), we have retained terms to order $\left(\frac{\hbar}{mc}\right)^2$ relative to the rest energy of the particle.

From the comparison of Eq. (3.2) with Eq. (3.1), one may recognize the physical significance of the new terms:

a) The term

$$-\frac{q\beta}{2mc}[(\mathbf{p}-\frac{q}{c}\mathbf{E}')\cdot\mathbf{E}'+\mathbf{E}'\cdot(\mathbf{p}-\frac{q}{c}\mathbf{E}')], \tag{3.3}$$

represents the interaction of a point charge q with the effective electric field $\mathbf{E}'$.

b) The term

$$\frac{q\hbar}{8m^2c^2}[\sigma\cdot(\mathbf{p}\times\mathbf{E}-\mathbf{E}\times\mathbf{p})], \tag{3.4}$$

may be rewritten in the form of

$$\frac{iq\hbar^2}{8m^2c^2}\sigma\cdot\frac{\partial\mathbf{B}}{\partial t} \tag{3.5}$$

with the aid of Eq. (2.9). Equation (3.5) represents the spin-orbit coupling associated with the magnetic moment. From Eqs. (3.3) and (3.5) we see that the effective Hamiltonian in our new representation Ψ' has a new feature and a clear physical meaning in the non-relativistic limit.

4. Conclusion

The transformation properties of the Dirac equation with the electromagnetic field have been discussed from the various points of view. The gauge-independent and path-independent unitary transformation has been introduced. And it has been found that the application of this transformation to the Dirac equation with the electromagnetic field is very useful to shed respectively a light on the scheme without potentials and the non-relativistic limit of the scheme. Especially it is stressed that the transformed Dirac Hamiltonian is of importance for contemplating the electromagnetic properties of the Dirac particle in terms of observables, such as the intensity of electric field **E** and the magnetic induction **B**.

Moreover, in the non-relativistic limit, it is of interest to investigate the behavior of the new transformed scheme. It is to be noted that the effective Hamiltonian of the scheme in the classical approximation has an interesting feature.

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